शिक्षक पेसागत विकास
स्रोत सामग्री (माध्यमिक तह)

हिसाव किताब

शैक्षिक जनशक्ति विकास केन्द्र
सानोठिमी, भक्तपुर
२०६८

प्रकाशक

नेपाल सरकार

शिक्षा मन्त्रालय

शैक्षिक जनशक्ति विकास केन्द्र

सानोठिमी , भक्तपुर

इमेल : **nced@moe.gov.np**

वेभ : **www.nced.gov.np**

भूमिका

शिक्षकको निरन्तर पेसागत विकासबाट नै प्रभावकारी शिक्षण सिकाइ व्यवस्थापन गर्न सकिन्छ । समग्र शैक्षिक कार्यक्रमको केन्द्र विन्दु बालक हो । सम्पूर्ण प्रयासहरूलाई विद्यालय र कक्षाकोठामा प्रतिबिम्बित गराई अपेक्षित उपलब्धि हासिल गराउने प्रमुख मानवीय शक्ति शिक्षक हो । त्यसैले शिक्षकको पेसागत विकास एक महत्त्वपूर्ण पक्ष हो ।

शिक्षकको निरन्तर पेसागत विकासको कार्यक्रमलाई सुनिश्चित गर्न शैक्षिक जनशक्ति विकास केन्द्रले दस दिने आवश्यकतामा आधारित मोड्युलर तालिम कार्यक्रम सञ्चालन गरेको छ । यो कार्यक्रम हालसम्मकै सबैभन्दा बढी विकेन्द्रीत र स्थानीयकरण (Decentralized and localized) गरिएको शिक्षक विकास कार्यक्रम हो । स्रोतकेन्द्र तहमा आधारित प्रस्तुत तालिम मोडालिटीबाट शिक्षक पेसागत सहयोग कार्यक्रम बढी सान्दर्भिक हुनको साथै शिक्षकका वास्तविक आवश्यकता र समस्यामा केन्द्रित हुने अपेक्षा गरिएको छ ।

शिक्षकको पेसागत विकास सञ्चालन गर्ने केन्द्रहरू (Training Hub) को क्षमता विकास गर्न विभिन्न प्रकारको सहायता कार्यक्रम सञ्चालनमा ल्याइएको छ । यीमध्ये प्रत्यक्ष मोडमा आधारित प्रशिक्षक-प्रशिक्षण कार्यक्रम, केन्द्र तथा क्षेत्रीय तहबाट स्थलगत सहायता र सामग्री सहयोग प्रमुख हुन । सामग्री सहयोग अन्तर्गत हरेक वर्ष शैक्षिक जनशक्ति विकास केन्द्रले प्रत्येक तालिम हवसम्म पुग्ने गरी स्रोतसामग्री तथा मोड्युल सामग्री उत्पादन गरी वितरण गर्ने व्यवस्था रही आएको छ । तालिम हवमा कार्यरत प्रशिक्षक र रोस्टर प्रशिक्षकहरूका लागि सन्दर्भ सामग्रीको रूपमा प्रस्तुत पुस्तकले मदत गर्ने छ भन्ने अपेक्षा गरिएको छ ।

शैक्षिक जनशक्ति विकास केन्द्र
सानोठिमी, भक्तपुर
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CONTENTS

1. The Greek Mathematics: Demonstrative Geometry

Characteristic of Greek Mathematics

In the 600 B.C. Mathematics was focused as a study and a science in the ancient Greek as a matter of course in China, India and Babylonia and to learn Geometry in Egypt. Thales, Pythagoras and Plato in Greek studied in Egypt and joined with Egypt culture Greek produced achievements at mathematics formed a term of now civilization accepting the Egypt culture. That is "Elements" of Euclid, "The Theory of conic sections" of Apollonius, "Arithmetica" of Diophantus and many research achievements of Archimedes. Many scholar represented as Aristotle. Plato focused only philosophy and mathematics. The story, Plato wrote "NO one knows Geometry, No admission" at the entrance to a hall, is famous. Euclid is known affected by Aristotle and plato. His "Elements" is the first arranged and systematized book logically and had been used as a textbook toward the end of the 1800's in Europe. This book showed the closed to the present mathematics toward 300 B.C. demonstrating a proposition from the axiom in the view of today, this had many defects, but this had affected after the that time. However, Greek mathematics was remarkable theoretically, but unremarkable in the field of number and calculus. The research in Algebra of Diophantus was remarkable. After that time, Europe had accepted arithmetic and Algebra from India and east countries until 900's.

Pythagorean mathematics

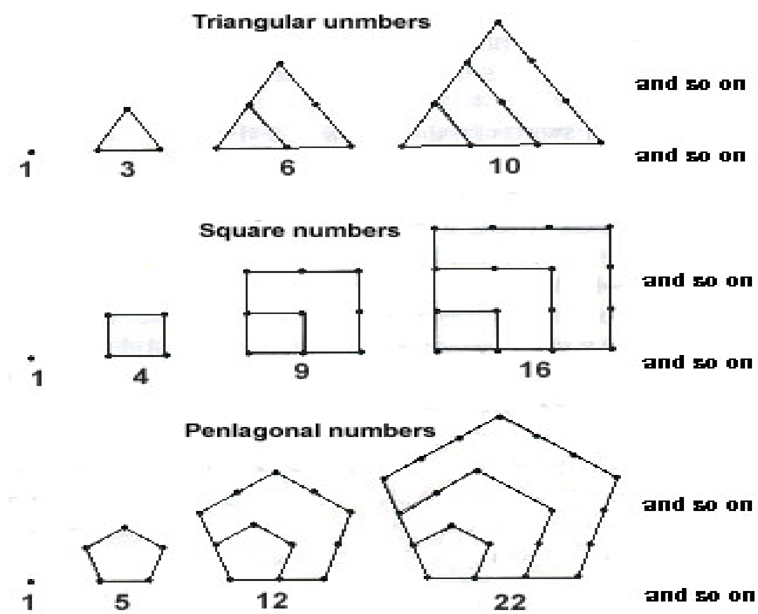
The Pythagorean philosophy rested on the assumption what whole number is the cause of the various qualities of man and matter. This led to an exaltation and study of number properties, and arithmetic (considered as the theory of numbers), along with geometry, music, and spherics (astronomy), constituted the fundamental liberal arts of the Pythagorean program of study.

Because Pythagoras' teaching was entirely oral, and because of the brotherhood's custom of referring all discoveries back to the revered founder, it is now difficult to know just which mathematical findings should be credited to Pythagoras himself and which to other members of the fraternity.

Pythagorean Arithmetic: Pythagoras and his followers, in conjunction with the fraternity's philosophy, took the first steps in the development of number theory, and at the same time laid much of the basis of future number mysticism. Amicable, or friendly, numbers. Two numbers are amicable number if each is the sum of the proper divisors of the other. For example, 284 and 220, constituting the pair ascribed to Pythagoras, are amicable. They are the perfect , deficient , and abundant numbers. A number is *perfect* if it is the sum of its proper divisors, *deficient* if it exceeds the sum of its proper divisors, and *abundant* if it is less than the sum of its proper divisors. So God created the world in six days, a perfect number, since 6= 1 + 2 + 3.

So people those times told fortunes with that number and they used an amulet to avert evils, the figurate numbers were found by the Pythagorean.

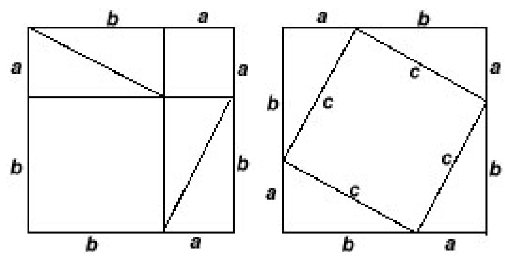
These numbers, considered as the number of dots in certain geometrical configurations, represent a link between geometry and arithmetic.



As a last and very remarkable discovery about numbers, made by the Pythagoreans, we might mention the dependence of musical intervals upon numerical ratios. The Pythagoreans found that for strings under the same tension, the lengths should be 2 to 1 for the octave 3 to 2 for the fifth, and 4 to 3 for the fourth. These results, the first recorded facts in mathematical physics, led the Pythagoreans to initiate the scientific study of musical scales.

Pythagorean Theorem and Discovery of Irrational Magnitudes: Pythagoras says that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs.

Since Pythagoras' time, many different proofs of the Pythagorean theorem have been supplied. In the second edition of his book, *The Pythagorean Proposition,* E.S. Loomis has collected and classified 370 demonstrations of this famous theorem.



Roughly saying the Pythagorean theorem is about width but acctually about the length of three sides to make a right triangle.

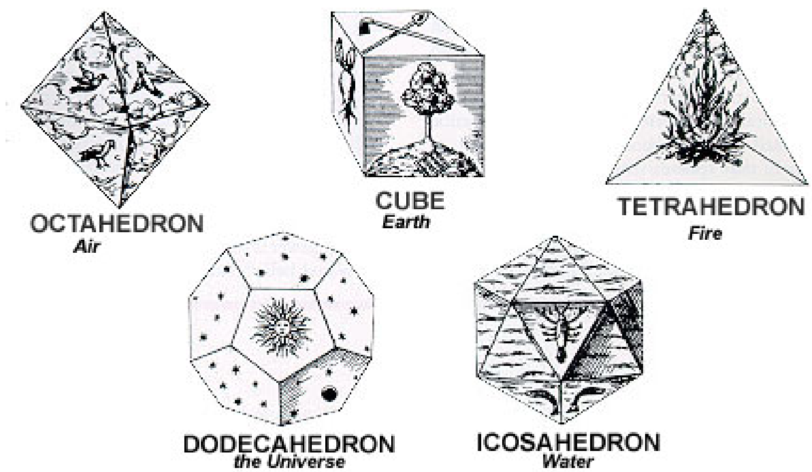
The problem of finding integers *a, b, c* that can represent the legs and hypotenuse of a right triangle. A triple of numbers of this sort is known as aPythagorean triple.

By this theorem there exist incommensurable line segments - that is, line segments having no common unit of measure. Thediscovery of irrational number is the milestone in mathematics history. But the discovery ran counter to the Pythogorean philosophy - 'everything is decided by integer.'

The discovery of the existence of irrational numers was surprising and disturbing to **the Pythagoreans.**

The Regular Solids: A polyhedron is said to be **regular** if its faces are congruent regular polygons and if its polyhedral angles are all congruent.

There is the tetrahedron with four triangular faces, the hexahedron, or cube, with six square faces, the octahedron with eight triangular faces, the dodecahedron with twelve pentagonal faces, and the icosahedron with twenty triangular faces . Plato mystically associates fire, earth, air, universe, and water to each regular solid.



The Three Famous Problems

The first three centuries of Greek mathematics, commencing with the initial efforts at demonstrative geometry by Thales about 600 B.C. and culminating with the remarkable *Elements* of Euclid about about 300 B.C. One can notice three important and distinct lines of development during the first 300 years of Greek mathematics. First, we have the development of the material that ultimately was organized into the *Elements*.

There is the development of notions connected with infinitesimals and with limit and summation processes. The third line of development is that of higher geometry, or the geometry of curves other than the circle and straight line, and of surfaces other than the sphere and plane. Curiously enough, most of this higher geometry originated in continued attempts to solvethree now famous construction problems. By virtue of this challenge, the development and creation of new mathematics were made.

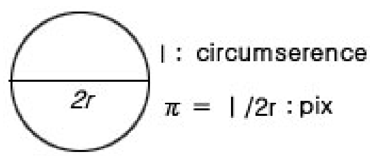
Duplication, Trisection, and Quadrature: The Greeks regarded logical thinking very highly. They considered hight system of knowledge as important: not practical value. Unexpectedly they couldn't solve easy problems Typical examples were duplication, trisection and quadrature.

1. *The duplication of the cube*, or the problem of constructing the edge of a cube having twice the volume of a given cube.
2. *The trisection of an angle,* or the problem of dividing a given arbitrary angle into three equal parts.
3. *The quadrature of the circle,* or the problem of constructing a square having an area equal to that of a given circle.

People should solved these three problems by using unmarked straightedges and compasses. The impossibility of the three constructions, under the self-imposed limitation that only the straightedge and compasses could be used, was not established until the nineteenth century, more than 2000 years after the problems were first conceived.

The energetic search for solutions to these three problems profoundly influenced Greek geometry and led to many fruitful discoveries, such as that of the conic sections, many cubic and quartic curves, and several transcendental curves. A much later outgrowth was the development of portions of the theory of equations concerning domains of rationality, algebraic numbers, and group theory.

A History of ¥ð: '¥ð' is used to calculate the area of a circle which is called ratio of circumference of circle to its diameter.



¥ð : the ratio of the circumference of a circle to its diameter

l : pheriphery of a circle

2r : diameter of a circle.

'¥ð'is fixed to any circles.

The man who used '¥ð' for the first time was Euler, Leonhard. If we, actually, want to calculate the area of a circle, we should know the value of '¥ð'.

Unable to reckon the accurate value of '¥ð' (nobody can do that), Archimedes got the approximate value of '¥ð'.

Starting from the regular inscribed and circumscribed six-sided polygons, Archimedes drew regular inscribed 96-sided polygons to the circle, and he drew regular circumscribed 96-sided polygons to it.

Then, the circumference of a circle is longer than that of the regular inscribed 96 - sided plygons and is smaller than that of the regular circumscribed 96 - sided polygons.Thus,

(circumference of an inscribed 96-side polygons) < *2¥ðr* (circumference of a circumscribed 96 - sided polygons)
3¡¤1/7 < ¥ð < 3¡¤10/71

This value is quite accurate ¡æ 3.14084 < ¥ð <3.142858

Archimedes used the approximate value of '¥ð' as 3.14.

Approximate value of '¥ð'.

Ahmes'(a.1650 B.C) Papyrus	¥ð ¡Ö 3.16
Arithmetic in Nine section	¥ð ¡Ö 3
Archimedes	¥ð ¡Ö 3.14
Tsu Chung - chih(430-501)	¥ð ¡Ö 3.1415929

Eucliod's <Elements>

Although Euclid was the author of at least ten works (fairly complete texts of five of these have come down to us), his reputation rests mainly on his *Elements*. It appears that this remarkable work immediately and completely superseded all previous *Elements*; in fact, no trace remains of the earlier efforts. As soon as the work appeared, it was accorded the highest respect, and from Euclid's successors on up to modern times, the mere citation of Euclid's book and proposition numbers was regarded as sufficient to identify a particular theorem or construction. No work, except the Bible, has been more widely used, edited, or studied, and probably no work has exercised a greater influcnce on scientific thinking. Over one thousand editions of Euclid's *Elements* have appeared since the first one printed in 1482; for more than two millennia, this work has dominated all teaching of geometry.

Contrary to widespread impressions, Euclid's *Elements* is not devoted to geometry alone, but contains much number theory and elementary (geometric) algebra. The work is composed of thirteen books with a total of 465 propositions. American high-school plane and solid geometry texts contain much of the material found in Books ¥°,¥²,¥³,¥µ,XI, and XII.

Certainly one of the greatest achievements of the early Greek mathematicians was the creation of the postulational form of thinking. In order to establish a statement in a deductive system, one must show that the statement is a necessary logical consequence of some previously established statements.

These, in their turn, must be established from some still more previously established statements, and so on. Since the chain cannot be continued backward indefinitely, one must, at the start, accept some finite body of statements without proof or else commit the unpardonable sin of circularity, by deducing statement A from statement B and then later B from A. These initially assumed statements are called the **postulates,** or **axioms,** of the discourse, and all other statements of the discourse must be logically implied by them. Where the statements of a discourse are so arranged, the discourse is said to be presented in postulational form.

So great was the impression made by the formal aspect of Euclid's *Elements* on following generations that the work became a model for rigorous mathematical demonstration.

It is not certain precisely what statements Euclid assumed for his postulates and axioms, nor, for that matter, exactly how many he had, for changes and additions were made by subsequent editors. There is fair evidence, however, that he adhered to the second distinction and that he probably assumed the equivalents of the following ten statements, five "axioms," or common notions, and five geometric "postulates":

A1 *Things that are equal to the same thing are also equal to one another.*

A2 *If equals be added to equals, the wholes are equal.*

A3 *If equals be subtracted from equals, the remainders are equal*

A4 *Things that coincide with one another are equl to one another.*

A5 *The whole is greater than the part.*

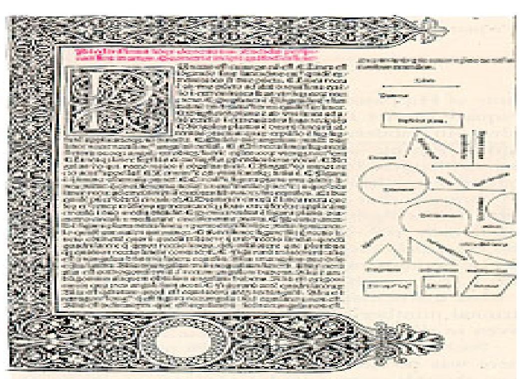
P1 *It is possible to draw a straight line from any point to any other point.*

P2 *It is possible to produce a finite straight line indefinitely in that straight line.*

P3 *It is possible to describe a circle with any point as center and with a radius qual to any to finite straight line drawn from the center.*

P4 *All right angles are equal to one another.*

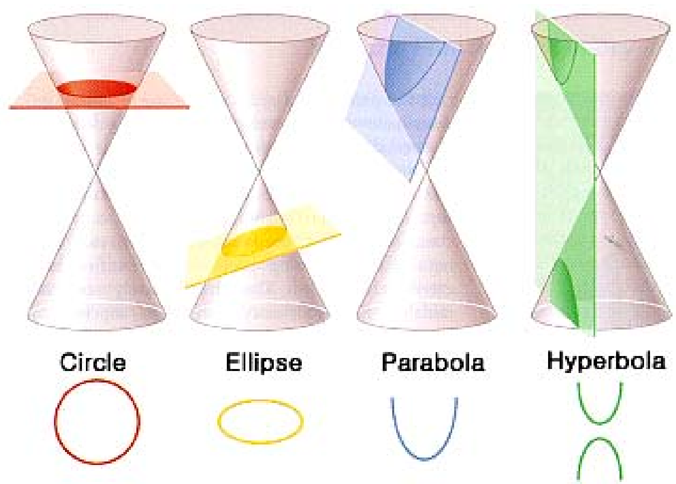
P5 *If a straight line intersects two straight lines so as to make the interior angles on one side of it together less than two right angles, these straight lines will intersect, if indefinitely produced, on the side on which are the angles which are together less than two right angles.*



A page from Euclid's *Elements.*

Greek Mathematics after Euclid

One of the greatest mathematicians of all time, and certainly the greatest of antiquity, was Archimedes, showed his typical strict arguments in calculating the area of a figure which was surrounded by parabola(curve) and chord (straight line). This way of reckoning provide the base of modern integral calculus. Great was his Knowledge about a circular cylinder and a sphere with Euclid and Archimedes in mathematics in 300 B.C. was a great mathematician Apollonius (ca. 200 B.C.) argued about <The theory of conic sections> which made him a great geometrician. He stated conic sections as cut stains from circular corns. These parts were omitted in <Elements> but Apollonius compiled many fields called ¡¸the theory of quadratic curve¡¹. This method reminds us of the analytic geometry.



Archimedes was killed by a roman soldier in 212 B.C. The Roman Empire conquered many city states in Greece and dominated the Mediterranean Sea. But the flower of science that is mathematics began to wither. Rome ruined Greek culture. In mathematics especially, Rome didn't obtain good results except quinary. The Roman Empire only assimilate and copy the conquered culture of Greece, Egypt and Carthage. Although the pursuit of learning weakened, Alexandria was the center of learning and culture then.

As trade was frequent between the West and the East, people came to need the art of navigation so they studied astronomy and trigonometry. Introduced was logistic system which represent angle today. Representative astronomers at those times were Aristarchus (280 B.C.) Eratosthenes and Hipparchus (150 B.C.) Eratosthenes, working at a library in Alexandria, computed the size of earth by measuring altitude of the sun on summer solstice.

Maybe more distinguished astronomer in Ancient Age, Hipparchus drew up the logistic system. He made a kind of table and it is called trigonometric function today and also studied spherical astronomy. <Syntaxis Mathematica> is maybe the best book about astronomy written by Claudius Ptolemy in Alexandria about 150 A.D. Arabians translated the book as <Almagest> which was regarded as a criterional book of astronomy from Copernicus to Kepler. Theoretical mathematics of Greece and practical mathematics of the orient coexisted at those times.

The representative mathematicians were Heron (250~150 B.C.) and Diophantus. The former is famous for its 'Heron's formula' referring to the area of a triangle. The latter is 'the father of algebra' who studied 'theory of numbers' and equation (primarily linear and quadratic) Pappus wrote <Mathematical collection> about Greek geometry. Hypatia, doughter of annotator Theon was also famous mathematician. As the Alexandrian School was burned by Arabians in 641. After this incident, the glorous and brilliant Greek mathematics disappeared in the darkness.

2. The Oriental Mathematics : Practical Arithmatic and Mensuration

Characteristic of Orient Mathematics
Babylonian Mathematics
Egyptian Mathematics
Marking of Number
The Egytian Hieroglyphic
The Babylonian Cuneiform
The Mayan Numeral System
The Roman Numeral System
The Hindu - Arabic Numeral Systern

Characteristic of Orient Mathematics

In the Nile in Africa, the Tigris and Euphrates in western Asia, the Indus and then the Ganges in south-central Asia, and the Hwang Ho and then the Yangtze in eastern Asia, there was ancient nations called the ancient 4-civilizations until 2000 B.C.

The major economic activities of the ancient nations was to manage their farmlands and to control their products. Thus, early mathematics can be said to have originated in certain areas of the ancient Orient (the world east of Greece) primarily as a practical science to assist in agriculture, engineering, and business pursuits, that is the initial emphasis of the early mathematics was on practical arithmetic and mensuration.

Algebra ultimately evolved from arithmetic and the beginnings of theoretical geometry grew out of mensuration.

However that in all ancient Oriental mathematics one cannot find even a single instance of what we today call a demonstration, and one cannot find the reason to get the answer so to speak 'Do it this way' then 'Get the answer'. That is, many difference from ancient Greek mathematics.

Mathematics was one of the essential parts in the ancient civilization. Today the only record is the Egypt and Babylonia's. Finally, the orient mathematics could not be developed because it was a 'living mathematics'.

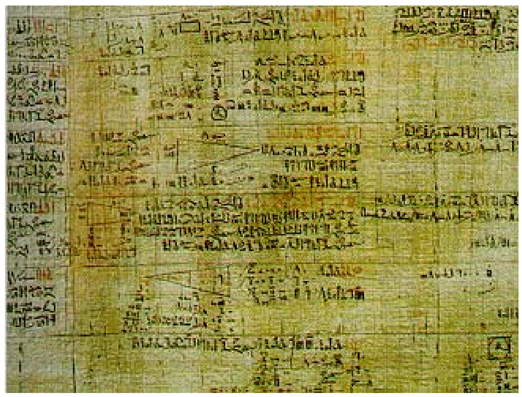
The Babylonians used imperishable baked clay tablets and the Egyptians used stone and papyrus, the latter fortunately being long lasting because of the unusually dry climate of the region. But the early Chinese and Indians used very perishable media like bark and bamboo. Thus, although a fair quantity of define information is now known about the science and the mathematics of ancient Babylonia and Egypt, very little is known with any degree of certainty about these studies in ancient China and India.

Babylonian Mathematics

The early Babylonians drew isosceles triangle on wet clay plates with needles. In this way, they made wedge-shaped letters. After making cuneiform they baked the plates to keep them for a long time. These plates were excavated at the Dynasty of King Hammurabi's era, about 1600 B.C. After deciphering the wedge-shaped letters, we can know that the Babylonians used very high system of calculation in commerce and agriculture with the sexagesimal positional system. Babylonian geometry is intimately related to practical mensuration. The chief feature of Babylonian geometry is algebraic character. Babylonians already knew the solution of quadratic equations and equations of second degree with two unknowns and they could also handle equations of the third and fourth degree. Thus the development of algebra quickened. We and undoubtedly owe to the ancient Babylonians our present division of the circumference of a circle into 360 equal parts.

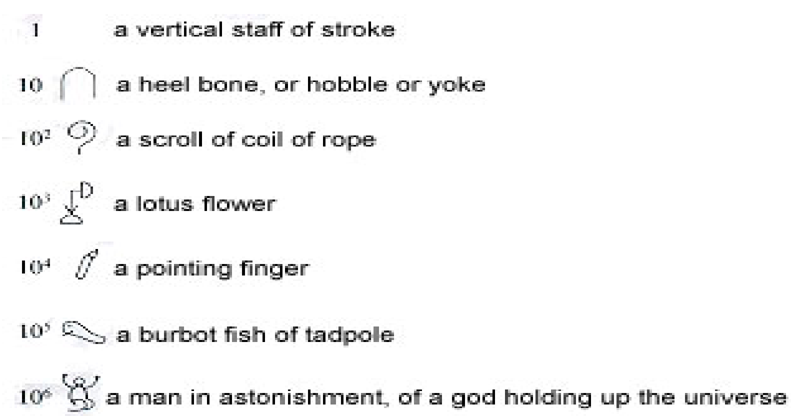
Egyptian Mathematics

Using a kind of reed,-papyrus- Egyptians made papers. About 1650 B.C. in 'Ahmes' Papyrus' which was written Ahmes, we can see how to calculate the fraction and the superficial measure of farmland. Ancient Egyptians say that the area of a circle is repeatedly taken as equal to that of the square of 8/9 of the diameter. They also extracted the volume of a right cylinder and the area of a triangle but they handled only a simple equation.

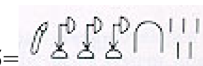


Marking of Number

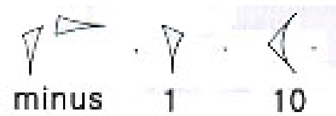
Probably the earliest way of keeping a count was by some simple tally method, employing the principle of one-to-one correspondence. In keeping a count on sheep, for example, one finger per sheep could be turned under. Counts could also be maintained by making collections of pebbles or sticks, by making scratches in the dirt or on a stone, by cutting notches in a piece of wood, or by tying knots in a string. As the way of counting, people should learn how to mark the numbers. Each nation, therefore, used its peculiar marking of numbers. The Egyptian Hieroglyphic: The Egyptian hieroglyphic numeral system is based on the scale of 10 and it was used about 3400 B.C.



Any number is now expressed by using these symbols additively, each symbol being repeated the required number of times. Thus,

$13015=1(10^4)+3(10^3)+1(10)+5=$ 

¡ÝThe babylonian Cuneiform : This was used from 2000 to 200 B.C. and it simplied the marking of numbers using the symbol '-'(minus)



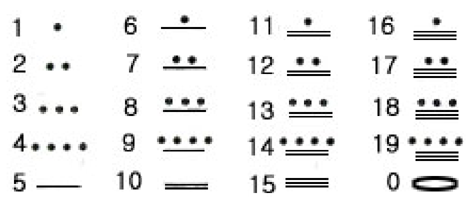
Thus, 38=40-2+ 

Sometime between 3000 and 2000 B.C., the ancient Baoyionians evolved a sexagesimal system employing the principle of position.

$524,551 = 2(60^3) + 25(60^2) + 42(60) + 31 =$ 

This method is the start of positional numberal system but the babylonians had difficulties because there was no '0'(zero) until about 300 B.C.

¡ÝThe Mayan Numeral System: This Mayan Numeral System has a symbol for '0' and is based on vigesimal. This is written very simply by dots and dashes.



An example of a larger number, written in the vertical Mayan manner, is shown below.

The rule of calculation for complex multiplication and division which are used in primary arithmetic was developed in late 15th century.

The reson why this rule was developed so late is there were no plenty of papers to record on (Chinese way of making papers was introduced in Europe after 12th century). They used abacus to overcome this difficulty.

Our present addition and subtraction patterns, along with the concepts of "carrying over" and "borrowing" may have originated in the processes for carrying out these operations on the abacus.

The Roman Numeral System: Numeral system was decimal system or quinary, the subtractive principle, in which a symbol for a smaller unit placed before a symbol for a larger unit means the difference of the two units, was used only sparingly in ancient and medieval times.

1	5	10	50	10©÷	500	10©ø
I	V	X	L	C	D	M

Thus, 1944=MDCCCCXXXXIIII
1994=MCMXLIV

This way disabled them from calculating multi-digits number so they used abacus.

The Hindu-Arabic Numeral System: 1,2,3,4,5,6,7,8,9,0

The Hindu-Arabic numeral system is named after the Hindus, who may have invented it, and after the Arabs, who transmitted it to western Europe.; The Persian mathematician al-Khowarizmi describes such a completed Hindu system used position value or 0(zero)in a book of A.D. 825.

It is not certain when this numeral system transmitted to Europe but this system was used all over the Europe about 13th century.

The dispute between the abacist and the algorist went on. Finally, the abacus disappeared in 18th century.

Our word *zero* probably comes from the Latinized form *zephirum* of the Arabic *sifr*, which in turn is a translation of the Hindu *sunya,* meaning "void" of "empty."

By virtue of the symbol of '0' the decimal system was established. And so we can use four operations more freely than ever.

3. The European Middle Ages Mathematics : Dark Ages of Monastic Mathematics

Characteristic of European Middle Ages Mathematics
Monastic Mathematics
Fibonacci and The 13th Century
The Antagonism of Commercial Against Monastic Mathematics
Non-European Mathematics
Indian Mathematics
Arabian Mathematics

Characteristic of European Middle Ages Mathematics:

Europe had accepted calculus and algebra from India and east counties until 900's. In India Aryabhata(475-553) wrote the numeration system and the astronomical observation theory on Aryabhattiya(449) in 600's. Arabic camber was invented in India. Italian Fibonacci introduced arabic number go Europe.

Monastic Mathematics:

We call the term the black Age from the middle of 400's to 1000's. In this times, the church controlled all the action and thinking of humans. Thus, there was no research of mathematics besides the research by tabbies of Catholic. Of the persons charitably credited with playing a role in the history of mathematics during the Dark Ages, we might mention the martyred Roman citizen Boethius, the British ecclesiastical scholars Bede and Alcuin, and the famous French scholar and churchman Gerbert, who became Pope Sylvester ¥±. The work of Boethius about arithmetic and geometry had been used as a textbook during many centuries.

Gerbert was known to spread Indian - Arabic number without 0 to Europe and also he was known to make an abacus, a terrestrial globe, a celestial globe and watch and establish the first school at France in Europe. After that time, mathematics in Europe started to progress in the end of the middle age and the early part of the Renessance (1100's-1400's). The knowledge in this times was based on not Greek but Islam mathematics. Arabic mathematics played an important part Greece (and India) with modern Europe. Europe in 1100's was the times of translation. The superior publication of Greek and Arabic mathematicians, Archimedes, Apollonius, Ptolemy, Menelaus and Al-Khowarizmi translated to Latin in Arabia.

Fibonacci and The 13th Century :

In the early part of 1200's Leonardo Fibonacci, the most talented mathematician in middle age, came on the stage. He was a man reconstructed mathematics of the middle age. He was interested in arithmetic in childhood influenced by his father, and he traveled Egypt, Sicily, Greece, Syria and had a chance to meet east and Arabic mathematics. Finally he came back home in 1202 and published the famous Liber abaci. Liber abaci shows to be influenced by algebra of Al-Khowarizmi and Abu Kamil. This book played an important part to introduce Indian - Arabic number to Europe, and had many problems. In this book, the following sequence is called Fibonacci's sequence.

$$1, 1, 2, 3, 5, \ldots, x, y, x+y, \ldots$$

The Antagonism of Commercial Against Monastic Mathematics :

Though Indian-Arabic a system of measuring by decimal notation spread among the merchants, mathematicians persevere in Roman a system of measuring against Indian - Arabic a system of measuring. They were churchmen. From this times the antagonism of progress against conservativeness appeared. This antagonism has been known the fight between the abacists and the algorists. The continuance of the antagonism proved that the algorists won finally but they waited until 1500's. The characteristic of the algorists was not only to calculus using 0 as a number without Indian - Arabic numeration system but also not to use abacus. At last, Indian calculation spread abroad become of the progress of commerce and industry con fronted by the period of prosperity.

Italy and Spain in 1400's and England, France and Germany in 17c used Indian - Arabic mathematics instead of Roman's. The greatest mathematician in 1300's was Nicole Oresme born at Normandy in 1323. He was a professor and became a bishop and died in 1382.

One of the books he wrote used a fraction and an exponent for the first time (not modern expression), the other expressed coordinates as a point. It become the origin of modern coordinates geometry.

This paper in the end of 1300's influenced Descartes and many Penessance mathematicians. Luca Pacioli (1445 - 1509), a Abby in Italy, wrote.<Summa de Arithmetica > This book contains many examples and commercial mathematics, especially bookkeeping by double entry.

Non-European Mathematics

We need to look over mathematics Arabia and India before moving to Middle and Modern Ages. That's why the two countries contributed to the development of the European middle ages mathematics.

Indian Mathematics:

Greek mathematicians were good at geometry but they were not at arithmetic and algebra because they didn't use signs. But Indian mathematicians actively used symbols and they made Indo-Arabian numbers. They also used decimal system. Indian mathematicians thought about the negative numbers for the first time and they made it a rule. For example, Brahmagupta divided numbers into two : property(positive number) and debt(negative number). But he didn't actually deal with 'negative numbers' freely as 'positive numbers.' He maybe thought that he could use 'negative numbers' in logical system not in practical. Bhaskara even said that 'negative numbers' were unable-to-get-acquainteo friends. But,surprisingly. Indo-Arabian numbers were as quite complete as people in other countries never dreamed it. The reasons why this kind of numbers were made and the art of calculation are as follows:

(1) They used very convenient tools for calculation
(Indians wrote numbers on a small blackboard with bamboo pen and white ink)

(2) It may sound paradoxical, but whey didn't know how to distinguish number from quality.

(3) Commerce developed in India earlier than other countries so they needed the art of calculation.

Although their achievements, they exposed some faults. Mathematics was for the nobilities so it tended to be games they, specially, expressed mathematics in the form of verse, which brought about despising the strict demonstration and inference. <Lilabati> written by Bhaskara is a good example. The name of the book is his dqughter's. It contained many meaningful contents but it is better known as a representative sanskrit literary works. It was Arabian who developed Indian mathematics' merit.

The field of algebra (equation) out of Europe-centered mathematics developed only in non-European countries. It was Europeans who used this Indo-Arabian mathematics but it developed so lively in Gupta Dynasty which had a great power in military, politics and culture from 4th to 12th century.

Arabian Mathematics

Arabians ruled parts of North Africa and Europe for 400 years since Mahomet (570?~632). They had new mathematics which was mixed Greek and Indian mathematics, which made Islam lead an important role in mathematics. Islamic mathematics, thus, became the starting point of modern European mathematics. When a slave state, Saracenic Empire, was formed, commerce and trade developed. People needed convenient and accurate art of calculation. Accurate maps were needed to Arabian merchant. Islamic ceremony(praying toward Mecca) had a great influence on the Arabian mathematics. Arabian merchants introduced Indian arithmetic and algebra into their commerce. On receipt of Greek study, Arabians praised it so much and they translated many Greek classics in Greek.

Finally, Arabians fused Greek logical geometry Indian arithmetic and algebra and they renewed them. Without Arabians' effort to preserve and study the Greek culture, important Greek achievements about mathematics would disappear. Arabian mathematics, thus, had a great role in the history of mathematics. Most people dealing with mathematics but Arabia were astronomers because commerce, administration, measurement, the way of making maps, astronomy and the calendar method were needed to calculate and survey the area of a land. So we can say that mathematics in Arabia served as a setoff for astronomy as in China and India.

Al-Khwarizmi was the most famous Arabian mathematician. He wrote two books about algebra and Indian numbers. When the two books were translated in Latin in 12th century, Europeans were quite influenced. 'Algorithm' today named after him means a certain process of calculation.

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4. The Sixteenth-Century Mathematics of Italy : Commercial Mathematics

Characteristic of The Sixteenth-Century Mathematics

Mathematics in 1400's-1500's in spite of the Renaissance revival was not developed after seventeenth century or Greek. The only thing focused was a solving of an equation of the third and fourth degree and symbolizing algebra of France. Though mathematics had a small change in Renaissance, it had powerful energy. However, the most important meaning is to make a modern mathematics start and to establish a tradition of European mathematics.

In summarizing the mathematical achievements of the sixteenth century, We can say that symbolic algebra was well started, computation with the Hindu-Arabic numerals became standardized, decimal fractions were developed, the cubic and quadratic equations were solved and the theory of equations generally advanced, negative numbers were becoming accepted trigonometry was perfected and systematized, and some excellent tables were computed. The stage was set for the remarkable strides of the next century.

Arrangement of The Symbols

Renaissant algebra started with necessity for commerce and arrangement of algebraic symbols.

Plus(+) and Minus(-) : These symbols appeared in a book about arithmetic written by John Widmann - Called father of arithmetic - for the first time in 1489. At first, these symbols expressed 'surplus', and 'insufficiency' but later it meant 'addition' and 'subtraction'

The symbol of minus (-) was in the book but the plus symbol(+) was not. Symbol, (+) was originated from Latin, 'et'(means 'or'), whereas we can't know the origin of symbol of minus(-).

Radical symbol(¡î) : Heinrich Schreiber Professor of Wien University used (+) and (-) to express addition and subtraction each in his book in 1521. His disciple, Christ off Rudolff used the radical symbol(¡î) including (+), (-), in his bool about algebra in 1525. He used simple the radical symbol (¡î) as (¥Ã) which might be from the first letter of root.

Equal Symbol(=) : This symbol appeared for the first time in<Whestone of wittle, 1557> known the first English algebraic book written by Robert Recorde (ca. 1510 ~ 1558). He said the reason why he adopted this symbol. "There is no other symbol than parallel lines(=) which means equality".

Division Symbol(¡À): Swiss mathematician Johann Heinrich Rahn used this symbol for the first time in his book <Teutsche Algebra> published in Zurich in 1659.

Decimal Symbol : Simon Stevin(1546~1620), a former technician, introduced this symbol for the first time.

Inequality Symbol(>,<) : These two Symbols were shown in a book published 10 years after English mathematician Thomas Harriot(1560~1621). After a century from his death, Pierre Bouguer started to use the symbols of ¡Ã and ¡Â.

Symbols of Multiplication(¡¿) and Difference(¡) : These symbols appeared in <Clavis mathematicae> (1631) written by English mathematician William Oughtred(1574~1660).

symbol of Letters : French mathematician Francois Viete (1540~1603) used letters to distinguish 'the known quantity' from 'the unknown'. He used consonants - as b,c,d,¡¦ - for 'the known quantity' and vowels - as, a, e, i, o and u - for 'the unknown' each.

But today, we use the fore part letters of alphabet - as, a,b,c,¡¦ - for 'the known quantity' and hind parts for 'the unknown' . This system started Rene Descartes (1596~1650).

The Introduction of these many mathematical symbols was closely related to the development of printing.

Cubic and Quadratic Equations

Probably the most spectacular mathematical achievement of the sixteenth century was the discovery, by Italian mathematicians, of the algebraic solution of cubic and quadratic equations. The story of this discovery, when told in its most colorful version, rivals any page ever written by Benvenuto Cellini. Briefly told, the facts seem to be these. About 1515, Scipione del Ferro (1465-1526), a professor of mathematics at the University of Bologna, solved algebraically the cubic equation $x^3 + mx = n$, probably basing his work on earlier Arabic sources. He did not publish his result but revealed the secret to his pupil Antonio Fior. Now about 1535, Nicolo Fontana of Brescia, commonly referred to as Tartaglia (the stammerer) because of a childhood injury that affected his speech, claimed to have discovered an algebraic solution of the cubic equation $x^3 + px^2 = n$. Believing this claim was a bluff, Fior challinged Targaglia to a public contest of solving cubic equations, whereupon the latter exerted himself and only a few days before the contest found an algebraic solution for cubics lacking a quadratic term. Entering the contest equipped to solve two types of cubic equations, whereas Fior could solve but one type, Tartaglia triumphed completely. Later Girolamo Cardano, an unprincipled genius who taught mathematics and practiced medicine in Milan, upon giving a solemn pledge of secredy, wheedled the key to the cubic form Tartaglia. In 1545, Cardano published his *Ars magna*, a great Latin treatise on algebra, at Neuremberg, Germany, and in it appeared Tartaglia's solution of the cubic. Tartaglia's vehement protests were met by Ludovico Ferrari, Cardano's most capable pupil, who argued that Cardano had received his information from del Ferro through a third party and accused Tartaglia of plagiarism from the same source. There ensued an acrimonious dispute from which Tartaglia was perhaps lucky to escape alive. Since the actors in the above drama seem not always to have had the highest regard for truth, one finds a number of variations in the details of the plot.

It was not long after the cubic had been solved that an algebraic solution was discovered for the general quadratic (or bi quadratic) equation. In 1540, the Italian mathematician Zuanne de Tonini da Coi proposed a problem to Cardano that led to quartic equation. Although Cardano was unable to solvce the equation, his pupil Ferrari succeeded, and Cardano had the pleasure of publishing this solution also in his *Ars magna*.

The representative mathematics of the 16th century is algebra originated in Arabia but it developed in Europe because commerce and calculation throve in there. Italian merchants and bankers, especially, needed now to calculate accurately.

Astronomy contribute to the development of mathematics and 'mathematician' meant 'astronomer' for some time. Nicolas Corpernicus (1473~1543), Polander, was the most distinguished astronomer who contributed so much to the development of mathematics. His theory about universe brought the improvement of trigonometry. He himself wrote a thesis on trigonometry.

Philosophies of Mathematics

There have arisen three main philosophies, or schools of thought, concerning the foundations of mathematics the so-called logistic, intuitionist, and formalist schools. Naturally, any modern philosophy of the foundations of mathematics must, somehow or other, cope with the present crisis in the foundations of mathematics

Russell and Whitehead's **LOGICISM:** The logistic thesis is that mathematics is a branch of logic. Rather than being just a tool of mathematics, logic becomes the progenitor of mathematics. All mathematical concepts are to be formulated in terms of logical concepts, and all theorems of mathematics are to be developed as theorems of logic; the distinction between mathematics and logic become merely one of practical convenience.

Alfred North Whitehead(1861-1947) and Bertrand Russell(1872-1970) deduced natural number system from hypothesis and set of axiom. They, therefore, identified many parts if mathematics with logic.

To avoid the contradictions of set theory. *Principia mathematica* employs a "theory of types."

Brower's **INTUITIONISM:** The intuitionist thesis is that mathematics is to be built solely by finite constructive methods in the intuitively given sequence of natural numbers, According to this view, then, at the very base of mathematics lies a primitive intuition, allied, no doubt, to our temporal sense of before and after, that allows us to conceive a single object, then one more, then one more, and so on endlessly.

For the intuitionists, a set cannot be thought of as a ready-made collection, but must be considered as a law by means of which the elements of the set can be constructed in a step-by-step fashion. This concept of set rules out the possibility of such contradictory sets as "the set of all sets."

Hilbert's **FORMALISM:** The formalist thesis is that mathematics is concerned with formal symbolic systems. In fact, mathematics is regarded as a collection of such abstract developments, in which the terms are mere symbols and the statements are formulas involving these symbols; the ultimate base of mathematics does not lie in logic but only in a collection of pre logical marks or symbols and in a set of operations with these marks. Since, from this point of view, mathematics is devoid of concrete content and contains only ideal symbolic elements, the establishment of the consistency of the various branches of mathematics becomes an important and necessary part of the formalist program.

Without such an accompanying consistency proof, the whole study is essentially senseless. In the formalist thesis, we have the axiomatic development of mathematics pushed to its extreme.

In his *Grundlagen der Geometrie*.(1899). Hilbert had sharpened the mathematical method from the material axiomatics of Euclid to the formal axiomatics of the present day. The formalist point of view was developed later by Hilbert to meet the crisis caused by the paradoxes of set theory and the challenge to classical mathematics caused by intuitionist criticism.

5. Constructivism as an approach of mathematics learning

Guiding Principles
Background
Some Critical Persepctives
Traditional vs Constructivism
The Teacher in a Constructivist Classroom
What does constructivism have to do with my classroom?
The Students in a Constructivist Classroom
Benefits of Constructivism
Impacts

Constructivism is a philosophy of learning founded on the premise that, by reflecting on our experiences, we construct our own understanding of the world we live in. Each of us generates our own "rules" and "mental models," which we use to make sense of our experiences. Learning, therefore, is simply the process of adjusting our mental models to accommodate new experiences.

Constructivism is basically a theory -- based on observation and scientific study -- about how people learn. It says that people construct their own understanding and knowledge of the world, through experiencing things and reflecting on those experiences. When we encounter something new, we have to reconcile it with our previous ideas and experience, maybe changing what we believe, or maybe discarding the new information as irrelevant. In any case, we are active creators of our own knowledge. To do this, we must ask questions, explore, and assess what we know.

In the classroom, the constructivist view of learning can point towards a number of different teaching practices. In the most general sense, it usually means encouraging students to use active techniques (experiments, real-world problem solving) to create more knowledge and then to reflect on and talk about what they are doing and how their understanding is changing. The teacher makes sure she understands the students' preexisting conceptions, and guides the activity to address them and then build on them.

Guiding Principles

1. Learning is a search for meaning. Therefore, learning must start with the issues around which students are actively trying to construct meaning.
2. Meaning requires understanding **wholes** as well as parts. And parts must be understood in the context of wholes. Therefore, the learning process focuses on primary concepts, not isolated facts.
3. In order to teach well, we must understand the mental models that students use to perceive the world and the assumptions they make to support those models.
4. The purpose of learning is for an individual to construct his or her own meaning, not just memorize the "right" answers and regurgitate someone else's meaning. Since education is inherently interdisciplinary, the only valuable way to measure learning is to make the assessment part of the learning process, ensuring it provides students with information on the quality of their learning.

Background

The concept of constructivism has roots in classical antiquity, going back to Socrates' dialogues with his followers, in which he asked directed questions that led his students to realize for themselves the weaknesses in their thinking. The Socratic dialogue is still an important tool in the way constructivist educators assess their students' learning and plan new learning experiences.

In this century, **Jean Piaget** and **John Dewey** developed theories of childhood development and education, what we now call Progressive Education, that led to the evolution of constructivism.

Piaget believed that humans learn through the construction of one logical structure after another. He also concluded that the logic of children and their modes of thinking are initially entirely different from those of adults. The implications of this theory and how he applied them have shaped the foundation for constructivist education.

Dewey called for education to be grounded in real experience. He wrote, "If you have doubts about how learning happens, engage in sustained inquiry: study, ponder, consider alternative possibilities and arrive at your belief grounded in evidence." Inquiry is a key part of constructivist learning.

Among the educators, philosophers, psychologists, and sociologists who have added new perspectives to constructivist learning theory and practice are **Lev Vygotsky** [3], **Jerome Bruner** [4], and **David Ausubel** [5].

Vygotsky introduced the social aspect of learning into constructivism. He defined the "zone of proximal learning," according to which students solve problems beyond their actual developmental level (but within their level of potential development) under adult guidance or in collaboration with more capable peers.

Bruner initiated curriculum change based on the notion that learning is an active, social process in which students construct new ideas or concepts based on their current knowledge.

Seymour Papert's [6] groundbreaking work in using computers to teach children has led to the widespread use of computer and information technology in constructivist environments.

Modern educators who have studied, written about, and practiced constructivist approaches to education include **John D. Bransford** [7], **Ernst von Glasersfeld** [8], **Eleanor Duckworth** [9], **George Forman** [10], **Roger Schank** [11], **Jacqueline Grennon Brooks** [12], and **Martin G. Brooks** [13]

Some Critical Perspectives

Constructivism has been criticized on various grounds. Some of the charges that critics level against it are:

1. It's elitist. Critics say that constructivism and other "progressive" educational theories have been most successful with children from privileged backgrounds who are fortunate in having outstanding teachers, committed parents, and rich home environments. They argue that disadvantaged children, lacking such resources, benefit more from more explicit instruction.

> *“In truth, progressivism didn't work with all 'privileged' kids, just those who had advantages at home or were smart enough to do discovery learning.”*
>
> — E.D. Hirsch

Social constructivism leads to "group think." Critics say the collaborative aspects of constructivist classrooms tend to produce a "tyranny of the majority," in which a few students' voices or interpretations dominate the group's conclusions, and dissenting students are forced to conform to the emerging consensus.

2. There is little hard evidence that constructivist methods work. Critics say that constructivists, by rejecting evaluation through testing and other external criteria, have made themselves unaccountable for their students' progress. Critics also say that studies of various kinds of instruction -- in particular **Project Follow Through** [1], a long-term government initiative -- have found that students in constructivist classrooms lag behind those in more traditional classrooms in basic skills.

Constructivists counter that in studies where children were compared on higher-order thinking skills, constructivist students seemed to outperform their peers.

Traditional vs Constructivism

As with many of the methods addressed in this series of workshops, in the constructivist classroom, the focus tends to shift from the teacher to the students. The classroom is no longer a place where the teacher ("expert") pours knowledge into passive students, who wait like empty vessels to be filled. In the constructivist model, the students are urged to be actively involved in their own process of learning. The teacher functions more as a facilitator who coaches, mediates, prompts, and helps students develop and assess their understanding, and thereby their learning. One of the teacher's biggest jobs becomes ASKING GOOD QUESTIONS.

And, in the constructivist classroom, both teacher and students think of knowledge not as inert factoids to be memorized, but as a dynamic, ever-changing view of the world we live in and the ability to successfully stretch and explore that view.

The chart below compares the traditional classroom to the constructivist one. You can see significant differences in basic assumptions about knowledge, students, and learning. (It's important, however, to bear in mind that constructivists acknowledge that students are constructing knowledge in traditional classrooms, too. It's really a matter of the emphasis being on the student, not on the instructor.)

Traditional	Constructivist
Curriculum begins with the parts of the whole. Emphasizes basic skills.	Curriculum emphasizes big concepts, beginning with the whole and expanding to include the parts.
Strict adherence to fixed curriculum is highly valued.	Pursuit of student questions and interests is valued.
Materials are primarily textbooks and workbooks.	Materials include primary sources of material and manipulative materials.
Learning is based on repetition.	Learning is interactive, building on what the student already knows.
Teachers disseminate information to	Teachers have a dialogue with students,

students; students are recipients of knowledge.	helping students construct their own knowledge.
Teacher's role is directive, rooted in authority.	Teacher's role is interactive, rooted in negotiation.
Assessment is through testing, correct answers.	Assessment includes student works, observations, and points of view, as well as tests. Process is as important as product.
Knowledge is seen as inert.	Knowledge is seen as dynamic, ever changing with our experiences.
Students work primarily alone.	Students work primarily in groups.

The Teacher in a Constructivist Classroom

Constructivist teachers encourage students to constantly assess how the activity is helping them gain understanding. By questioning themselves and their strategies, students in the constructivist classroom ideally become "expert learners." This gives them ever-broadening tools to keep learning. With a well-planned classroom environment, the students learn HOW TO LEARN.

You might look at it as a spiral. When they continuously reflect on their experiences, students find their ideas gaining in complexity and power, and they develop increasingly strong abilities to integrate new information. One of the teacher's main roles becomes to encourage this learning and reflection process.

Contrary to criticisms by some (conservative/traditional) educators, constructivism does not dismiss the active role of the teacher or the value of expert knowledge. Constructivism modifies that role, so that teachers help students to construct knowledge rather than to reproduce a series of facts. The constructivist teacher provides tools such as problem-solving and inquiry-based learning activities with which students formulate and test their ideas, draw conclusions and inferences, and pool and convey their knowledge in a collaborative learning environment. Constructivism transforms the student from a passive recipient of information to an active participant in the learning process. Always guided by the teacher, students construct their knowledge actively rather than just mechanically ingesting knowledge from the teacher or the textbook.

Constructivism is also often misconstrued as a learning theory that compels students to "reinvent the wheel." In fact, constructivism taps into and triggers the student's innate curiosity about the world and how things work. Students do not reinvent the wheel but, rather, attempt to understand how it turns, how it functions. They become engaged by applying their existing knowledge and real-world experience, learning to hypothesize, testing their theories, and ultimately drawing conclusions from their findings.

The best way for you to really understand what constructivism is and what it means in your classroom is by seeing examples of it at work, speaking with others about it, and trying it yourself. As you progress through each segment of this workshop, keep in mind questions or ideas to share with your colleagues.

What does constructivism have to do with my classroom?

As is the case with many of the current/popular paradigms, you're probably already using the constructivist approach to some degree. Constructivist teachers pose questions and problems, then guide students to help them find their own answers. They use many techniques in the teaching process. For example, they may:

- prompt students to formulate their own questions (inquiry)
- allow multiple interpretations and expressions of learning (multiple intelligences)
- encourage group work and the use of peers as resources (collaborative learning)

More information on the above processes is covered in other workshops in this series. For now, it's important to realize that the constructivist approach borrows from many other practices in the pursuit of its primary goal: helping students learn HOW TO LEARN.

The Students in a Constructivist Classroom

Students are not blank slates upon which knowledge is etched. They come to learning situations with already formulated knowledge, ideas, and understandings. This previous knowledge is the raw material for the new knowledge they will create.

Example: An elementary school teacher presents a class problem to measure the length of the "Mayflower." Rather than starting the problem by introducing the ruler, the teacher allows students to reflect and to construct their own methods of measurement. One student offers the knowledge that a doctor said he is four feet tall. Another says she knows horses are measured in "hands." The students discuss these and other methods they have heard about, and decide on one to apply to the problem.

The student is the person who creates new understanding for him/herself. The teacher coaches, moderates, suggest, but allow the students room to experiment, ask questions, try things that don't work. Learning activities require the students' full participation (like hands-on experiments). An important part of the learning process is that students reflect on, and talk about, their activities. Students also help set their own goals and means of assessment.

The constructivist classroom relies heavily on collaboration among students. There are many reasons why collaboration contributes to learning. The main reason it is used so much in constructivism is that students learn about learning not only from themselves, but also from their peers. When students review and reflect on their learning processes together, they can pick up strategies and methods from one another.

The main activity in a constructivist classroom is solving problems. Students use inquiry methods to ask questions, investigate a topic, and use a variety of resources to find solutions and answers. As students explore the topic, they draw conclusions, and, as exploration continues, they revisit those conclusions. Exploration of questions leads to more questions.

Students have ideas that they may later see were invalid, incorrect, or insufficient to explain new experiences. These ideas are temporary steps in the integration of knowledge. For instance, a child may believe that all trees lose their leaves in the fall, until she visits an evergreen forest. Constructivist teaching takes into account students' current conceptions and builds from there.

What happens when a student gets a new piece of information? The constructivist model says that the student compares the information to the knowledge and understanding he/she already has, and one of three things can occur:

- The new information matches up with his previous knowledge pretty well (it's **consonant** with the previous knowledge), so the student adds it to his understanding. It may take some work, but it's just a matter of finding the right fit, as with a puzzle piece.
- The information doesn't match previous knowledge (it's **dissonant**). The student has to change her previous understanding to find a fit for the information. This can be harder work.
- The information doesn't match previous knowledge, and it is **ignored**. Rejected bits of information may just not be absorbed by the student. Or they may float around, waiting for the day when the student's understanding has developed and permits a fit.

Benefits of Constructivism

- Children learn more, and enjoy learning more when they are actively involved, rather than passive listeners.

- Education works best when it concentrates on thinking and understanding, rather than on rote memorization. Constructivism concentrates on learning how to think and understand.
- Constructivist learning is transferable. In constructivist classrooms, students create organizing principles that they can take with them to other learning settings.
- Constructivism gives students ownership of what they learn, since learning is based on students' questions and explorations, and often the students have a hand in designing the assessments as well. Constructivist assessment engages the students' initiatives and personal investments in their journals, research reports, physical models, and artistic representations. Engaging the creative instincts develops students' abilities to express knowledge through a variety of ways. The students are also more likely to retain and transfer the new knowledge to real life.
- By grounding learning activities in an authentic, real-world context, constructivism stimulates and engages students. Students in constructivist classrooms learn to question things and to apply their natural curiosity to the world.
- Constructivism promotes social and communication skills by creating a classroom environment that emphasizes collaboration and exchange of ideas. Students must learn how to articulate their ideas clearly as well as to collaborate on tasks effectively by sharing in group projects. Students must therefore exchange ideas and so must learn to "negotiate" with others and to evaluate their contributions in a socially acceptable manner. This is essential to success in the real world, since they will always be exposed to a variety of experiences in which they will have to cooperate and navigate among the ideas of others.

Impacts

Curriculum–Constructivism calls for the elimination of a standardized curriculum. Instead, it promotes using curricula customized to the students' prior knowledge. Also, it emphasizes hands-on problem solving.

Instruction–Under the theory of constructivism, educators focus on making connections between facts and fostering new understanding in students. Instructors tailor their teaching strategies to student responses and encourage students to analyze, interpret, and predict information. Teachers also rely heavily on open-ended questions and promote extensive dialogue among students.

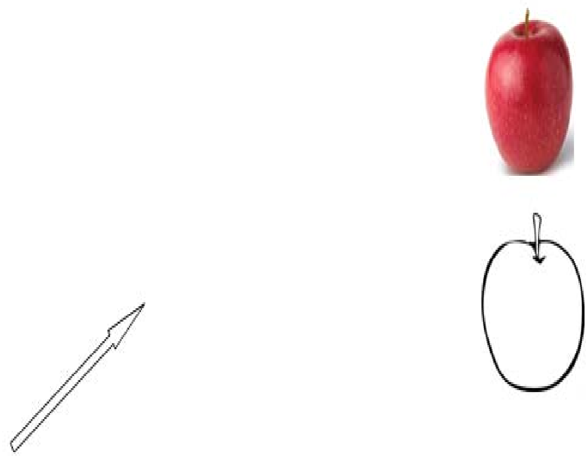
Assessment–Constructivism calls for the elimination of grades and standardized testing. Instead, assessment becomes part of the learning process so that students play a larger role in judging their own progress.

Constructivist ways of teaching/learning activities (some examples)

Relating Two Quantities

Overview

There are many quantities that seem to be related with each other. This lesson deels with situations suggesting relationships between two quantities. The participants will discover the relationship between these quantities, some basic ideas of functional relationship are reviewed and further emphasized. Further, finding how the quantities are related in these addition, the lesson demonstrates how a numerical solution can be used to arrived at a generalized solution. Connection among the different solutions is also emphasized.



Instructional procedures

Introductory activity

Study this diagram/picture.

The arrow is shot at the same time the star apple is falling.

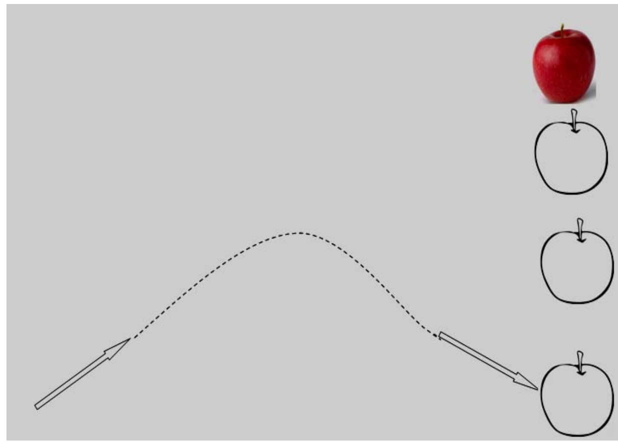
Will the arrow likely to hit the star apple? Why?

Expected answer

Yes,the arrow will likely hit the star apple.

The arrow reaches a maximum height then goes down due to gravity. The star apple also falls down due to gravity. Since the arrow and the star apple are both falling due to gravity, it is likely to happen, that the arrow will hit the star apple.

Here is a result of an experiment that follows the plight of an object falling due to gravity. Explain that distance is measured from the time the object is dropped.



Time (sec)	0	0.4	0.6	0.8	1.0
Distance (m)	0	0.8	1.8	3.2	5

What do you observe ?

Expected answers

a. The distance changes as time changes

b. For each value of time there corresponds a single value of height

c. As time increases, the distance also increases.

Predict the height when the time is 1.2 second?

Show your solution

Possible answers

Method 1. Looking for a pattern.

To lead the participants to this solution, let then observe the given values.

0.2 0.2 0.2 0.2

Time (sec)	0	0.4	0.6	0.8	1.0	1.2
Distance (m)	0	0.8	1.8	3.2	5	7.2

1 1.4 1.8 2.2

Method 2. Graphing

Using the graph, the distance that corresponds to ½ sec is approximately 7.2 m

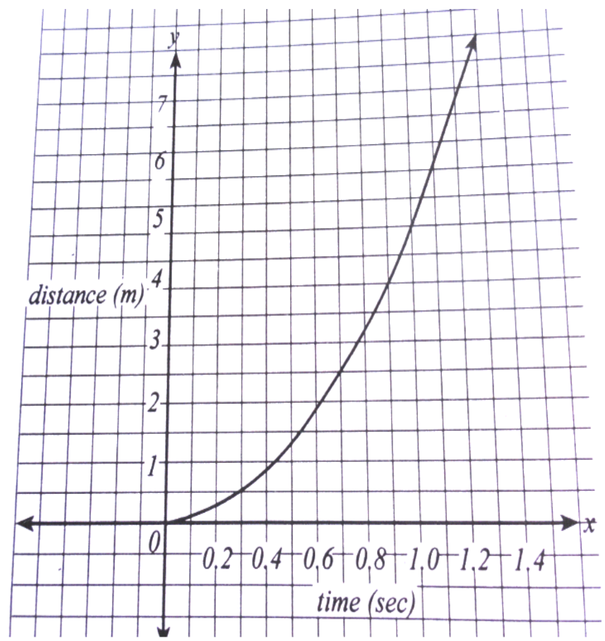
Method 3. Using an equation

The formula for distance traveled by a freely object is given by :

$Y=1/2\ gx^2$, where y is the distance traveled by a freely falling body

X is the time the body is in motion

g is the gravitational attraction on the body



For the participants be able to think of this solution, let them recall the formula for the distance traveled by a freely falling body.

In the metric system of measurement,the value of g is 9.8 m/sec^2.

Hence, y=5x2

So, when time = 1.2 sec

$Y=(5m/sec^2)(1.2\ sec\ )^2$

= 7.2 m

You have noticed that in the situation we have considered, time and distance the object falls vary with each other.

Now, can you cite some situations in daily life which involve quantities that vary with each others ? Describe how these quantities are related using arrow diagram, ordered pairs in tables, rules expressed in words and equations.

Possible answers

a. Buying an item

Arrow diagram

No. of ballpens cost (p)

1 6 ⟶

2 12 ⟶

3 18

. .

. .

Answers may vary. The answers suggested here are just two of the many possible answers.

Ordered pairs in table

X(no.of ballpens)	0	1	2	3	………
Y (cost in Rs)	0	6	12	18	…….

Rules expressed in words

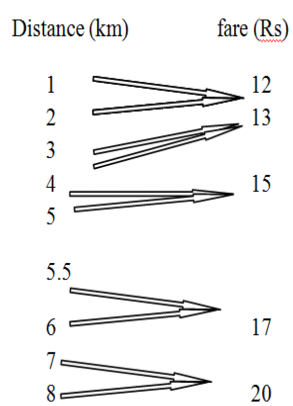
The cost of ballpens bought is equal to 6 Rs, multiplied by the number of ballpens bought

Equation

Cost = p6 multiplied by the no.of ballpens

Y= 6x

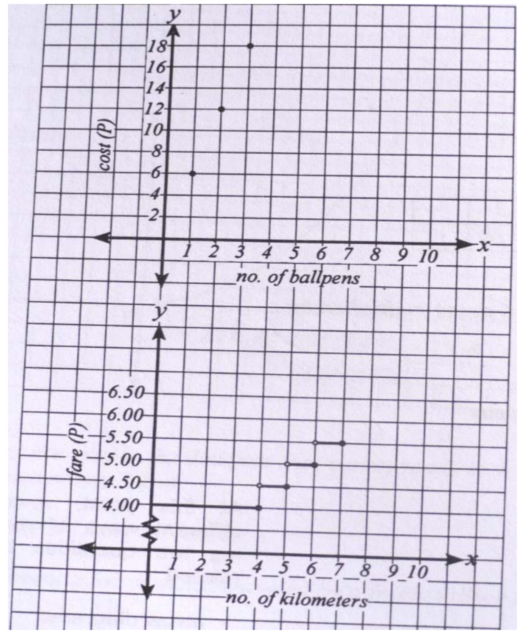
b. Riding in a bus



At this point, review the different ways of representing two quantities that are related

- Arrow diagrams
- Ordered pairs in tables or lists
- Rules expressed in words

- Equations
- Graphs



Mention that although an arrow diagram clearly pictures the correspondence of the values. It has its limitation. In cases where more pairs are involved, an arrow diagram is not advisable to use. Graphs can be used to picture the relationship between the quantities.

Recall at this point, that a relationship describing how one quantity varies with the other quantity is a function . Do not forget to emphasize however, that not all relationships are functions. If you represent the two examples you have given by a graph, how would the graphs look ?

Call their attention regarding the correspondence of values.

For the first example they gave and the situation previously considered, one value of one quantity corresponds to only one value of the other quantity. This correspondence is called one to one. A one to one correspondence describes a function.

Intervention Strategies for Mathematics Teachers

Intervention has become an important way for teachers to ensure that all students succeed in today's high stakes testing environment. Helping students who are struggling with mathematics requires teachers to choose an appropriate time and strategy for the intervention. Without a systematic approach, this can be a challenge for teachers who have multiple students in need of help.

Following are some easy strategies to help you **identify** students who may benefit from intervention, and **address** the needs of those students.

Step One: Identify

Use the following easy and effective strategies to help you identify students who may be struggling and who may benefit from intervention strategies.

- **Use Formal and Informal Assessments**
  No single instructional strategy is more important than effective, appropriate, and informative assessment. It is critical that teachers are well-informed about their

students' understanding and mastery of content. But assessment should also be handled with restraint—too much testing may produce students who are weary and overwhelmed. Use the following techniques when assessing your students.

- Use informal techniques frequently during regular class time to gauge student understanding.
- Use questioning that focuses on student thinking and reasoning to help you monitor your students.
- Incorporate writing activities and group work to observe student thinking and identify misconceptions and gaps in understanding.
- Have students illustrate concepts using drawings, graphs, and models.

- **Integrate Warm-Up Activities** The use of quick warm-up activities in class can be beneficial for several reasons. One of the most common reasons students may need intervention is that they have not fully mastered prerequisites. You can use warm-up activities to review prerequisites and to gauge student mastery. Begin your lessons by having your students complete several problems that cover prerequisites. This technique will also give you time to circulate among your students and have quiet one-on-one
- conversations. These discussions can be used as valuable informal assessment opportunities.

Warm-Up Activity
For a unit on solving systems of linear inequalities, ask students to solve several inequalities as a warm-up activity. Then have your students graph a few inequalities.

- **Write to Learn**

  Having students write in math class can help you identify areas of misunderstanding and gaps in understanding. Begin your instructional units by having your students write explanations of several key prerequisites. Students may feel more comfortable writing and may be more apt to expose their weaknesses in their writing. This can be especially true for struggling students who may be inclined to stay quiet during discussions. Use math journals to have students record the steps they undertook to solve a problem. You can use their explanations as a form of error analysis to help you identify gaps in understanding.

- **Assign Application Problems**
  Make sure that you utilize a variety of techniques to gauge depth of understanding in your students. Some students who have a cursory understanding of a topic may be able to perform relatively well on standard assessment questions. However, the lack of mastery of a concept can be illuminated via application problems. This exercise can be especially important prior to moving on to a new concept. An application problem can identify students who have not thoroughly mastered a concept and who will likely require intervention if they move on to a new concept too soon.

Step Two: Address the Issues
Using the following instructional strategies to help you address the needs of your students.

- **Use Small Groups or Student Pairs**
  Having your students work in small groups or in student pairs is a beneficial instructional strategy for struggling students. Students who need intervention may be insecure about their abilities and consequently unmotivated. Small groups or student pairs can be less intimidating for struggling students. Students may be more likely to ask questions and admit confusion when working in small groups or with another student.

  Students can also benefit from explanations from fellow students. Often these explanations can make more sense to a student than one offered from an instructor. This instructional strategy can enable teachers to spend time listening to and observing students as they work on assignments.

  The grouping of students should be carefully thought out ahead of time to best address the needs of struggling students. For many cooperative group activities, random assignments are fine, but in the case of students in need of intervention, you will want to form groups or pairs that will be conducive to discussion and support.

- **Differentiate Instruction**
  When it comes to addressing students who need intervention, differentiated strategies may improve learning. Many students who need intervention struggle to learn concepts because they may not be able to grasp abstract concepts. Vary your instructional techniques to best address the learning styles of your struggling students. Some students may not understand a concept when illustrated symbolically, but may be able to understand it when it is illustrated concretely, either via models, manipulatives, or technology. The more varied instructional strategies you incorporate into your lessons, the more likely you will be able to reach all students.

- **Incorporate Multiple Representations**
  Many middle and upper grade students require intervention because they are not able to grasp the abstract concepts of higher levels of mathematics. The use of multiple representations can help address these needs. When introducing a new concept, use as many representations of the concept as you can: use manipulatives and models, real-life examples, technology, and symbolic representations.

Try This
For a lesson on parallel and perpendicular lines, use the following multiple representations: o Show examples of parallel and perpendicular lines in architecture and art. o Give students straws, sticks, threads to model these lines. o Use dynamic geometry, such as the Geometer's Sketchpad software, to demonstrate parallel and perpendicular lines. o Have your students record in their math journals several examples of lines that can be found in the world around them.

- **Emphasize Real-Life Applications**
  Help students see the value and application of the mathematics they are studying by presenting as many real-life applications as you can. By relating a math topic to something relevant in a student's life, you can help increase a student's interest in the topic, and help make mathematics more meaningful. This can be especially beneficial for struggling students who may not be able to see how the math they are studying has any relevance to their daily lives. Many real-life applications of mathematics can make the content more interesting to struggling students. By increasing their interest, you can help increase their motivation.

- **Learn About Tutoring Options**
  In addition to these instructional strategies, you should also learn about tutoring options that may be available to your students.
    - Does your school have an after-school tutoring program?
    - Are there low-cost tutoring centers near your school?
    - Are there any mentoring programs available for your students?

Know the tutoring options that are available for the students who may need something extra to help address their needs.

- **Consider Seating Arrangements**
  Sometimes intervention can be as simple as where your students sit in your classroom. Sometimes physical placement can get overlooked once students reach the middle and upper grades. Strategically seat your struggling students in the best location in your classroom, where they feel most comfortable, can focus on the lesson, and may benefit from a helpful student peer nearby.

This article was contributed by Heidi Janzen, a former classroom teacher and mathematics specialist. She now works as an educational consultant in the areas of professional development, curriculum, standards, and assessment.

6. Games for teaching mathematics

Probability based Games

Probability is an area of mathematics that often doesn't get its fair share of attention in elementary classrooms. Here are some activities to get you started that involve students in thinking about probability ideaswhile also providing practice with mental addition, experience with strategic thinking, and the opportunity to relate multiplication and geometry. All activities are adapted from Marilyn Burns's *About Teaching Mathematics* (Math Solutions Publications, 1992).

The Game of Pig (Grades 3–8)

Math concepts: This game for two or more players gives students practice with mental addition and experience with thinking strategically.

The object: to be the first to score 100 points or more.

How to play: Players take turns rolling two dice and following these rules:

1. On a turn, a player may roll the dice as many times as he or she wants, mentally keeping a running total of the sums that come up. When the player stops rolling, he or she records the total and adds it to the scores from previous rounds.
2. But, if a 1 comes up on one of the dice before the player decides to stop rolling, the player scores 0 for that round and it's the next player's turn.
3. Even worse, if a 1 comes up on both dice, not only does the turn end, but the player's entire accumulated total returns to 0.

After students have had the chance to play the game for several days, have a class discussion about the strategies they used. You may want to list their ideas and have them test different strategies against each other to try and determine the best way to play.

Two-Dice Sums (Grades 1–8)

Math concepts: Students of all ages can play this game, as long as they're able to add the numbers that come up on two dice. While younger children benefit from the practice of adding, older students have the opportunity to think about the probability of the sums from rolling two dice.

The object: to remove all the counters in the fewest rolls possible.

How to play: Two or more players can play. Each player needs 11 counters, a game strip that lists the numbers from 2 to 12 spaced far enough apart so the counters can fit on top of each number, and a recording sheet. Here are the rules for playing:

1. Each player arranges 11 counters on the game strip and records the arrangement.
2. Once the counters are arranged, players take turns rolling the dice.
3. For each roll, all players can remove one counter if it is on the sum rolled. Players keep track of the number of rolls of the dice it takes to clear their game board.

After students have had the chance to play the game for several days or so, have a class discussion about the different ways they arranged the counters and the number of rolls it took. Have them write about the arrangements that are best for removing the counters in the fewest number of rolls. For an extension, try Which Number Wins?

Which Number Wins? (Grades 1–8)

Math concepts: In this individual activity, students roll two dice and record the results. Make a recording sheet that is an 11 x 12 block grid with the numbers 2 through 12 across the top. While young children gain practice with addition facts, older children can examine the data, compare results with other classmates, and think about why some sums are more likely than others. To do the activity, students need two dice and a recording sheet.

The object: to roll the dice and record the number fact in the correct column, stopping when one number gets to the finish line.

How to play: Post a class chart that lists the numbers from 2 to 12 and have students make a tally mark to show the winning sum. Have each child do the experiment at least twice.

After you've collected the data, discuss with the class why it seems that some sums "win" more than others. Young children may not be able to explain it, but older students often figure out that there is only one way to get the sums of 2 and 12, and six ways to get a sum of 7.

After discussing the data, return to the game of Two-Dice Sums and see if students revise their strategies. You may want to ask students to write about the game and the likelihood of two-dice sums.

How Long? How Many? (Grades 3–5)

Math skills: This two-person game involves probability and strategy, and gives children experience with multiplication in a geometric context.

The object: to make rectangular arrays with Cuisenaire Rods and place them on 10-by-10-centimeter grids until no more space is available. The game encourages students to think strategically as they consider where to place their rectangles to avoid being blocked.

How to play: students need Cuisenaire Rods, one die, and a grid sheet for each (Make a 10cm x 10cm grid. Also leave space for students to record how many of their squares are covered and uncovered.) The rules are:

1. On his or her turn, a player rolls the die twice to determine which Cuisenaire Rods to take. The first roll tells "how long" a rod to use. The second roll tells "how many" rods to take.
2. Players arrange their rods into a rectangle, place it on their grid, and trace it. They write the multiplication sentence inside.
3. The game is over when one player can't place a rectangle because there's no room on the grid. Then players figure out how many of their squares are covered and how many are uncovered and check each other's answers.

After students have had experience playing the game, talk with them about strategies for placing rectangles and figuring out their final scores.

(Adapted from *Instructor*, April 1994.)

Rescue Mission Game

Students play a game to learn about the four forces of flight: lift, drag, thrust, and weight. Before playing the game, students conduct a probability experiment with spinners and record their results in tally tables and bar graphs. They then use their findings to select spinners with the greatest probability of helping them win the game. In a portion of the game, students use ordered pairs to plot points on the coordinate plane to show their flight path.

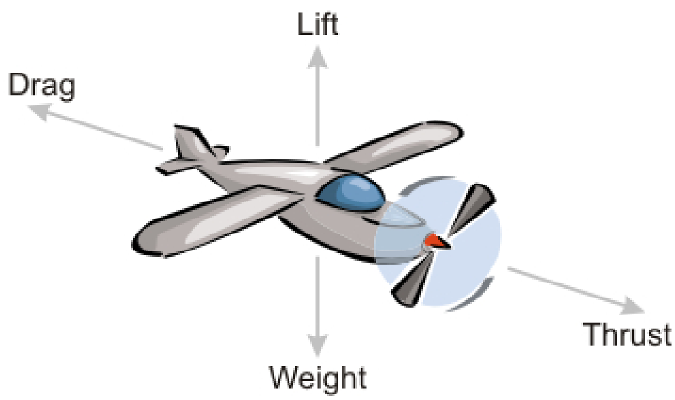
This lesson was adapted from *Travel in the Solar System* in Mission Mathematics II: Grades 3-5, a NASA/NCTM project, NCTM 1997.

Background Information

When we look at large airliners and helicopters, it seems impossible for such huge objects to lift off the ground and fly. Flight is possible because of four forces (pushes or pulls) that act on the aircraft.

Two of the forces are lift and weight. Lift is the upward force that works against the force of weight, the force that holds the aircraft down. Lift is created by the effect of airflow over and under the wings of airplanes or the blades of helicopters. Wings are usually thicker on the front edge and thinner on the back edge. This shape allows the air moving over the wing to move faster, and consequently, to have less pressure. The air moving under the wing reveals more slowly and results in more pressure pushing up on the wing. Thus, the force pushing up on the wing is greater that the force pushing down.

The other two forces of flight are thrust and drag. Thrust is the push or the pull forward that causes aircraft to move. The engines create thrust. Drag is an opposite force that slows the aircraft. Drag is caused by the surfaces of the aircraft that interrupt or deflect the smooth airflow around the aircraft. Some things that affect the amount of drag are the flaps; the ailerons; and the size, shape, and position of the wings.



Getting Started

As you introduce or review the forces of flight, ask questions to focus students' attention on a diagram with arrows to show the direction associated with each of the four forces. For example, ask which force pulls things to the ground.

As you present the Rescue Mission Game activity sheet, introduce the game students will be playing. They are pilots of rescue helicopters. Their mission is to fly their helicopters to the top of a mountain to rescue lost hikers.

As you discuss the Rescue Mission Game activity sheet, explain how the spinner determines in which direction to move. For example, if the pointer lands on Lift, students move their helicopter up one space. Ask them the following questions:

In which direction they should move if it lands on Drag. [Left]

On Thrust? [Right]

On Weight? [Down]

When students cannot move in the direction indicated by the spinner, they stay in the same position for that turn.

Show students the starting point of the game and ask them to think about the flight path for their mission. Ask students the following questions:

In which directions they must go to reach the mountaintop. [Up and to the right]

Which forces will be most helpful. [Lift and thrust] Why?

Developing the Activity

Part 1: Getting Ready for the Mission

As you discuss the Rescue Mission Game activity sheet, explain that since the lost hikers are cold and hungry, the pilot needs to get to the top of the mountain quickly. Ask students to compare the spinners.

Class Conversation

Use the following questions to guide the class conversation:

- How are the spinners alike? How are they different?
- Which spinner do you think will help your helicopter get to the mountaintop the fastest? Why do you think so?
- How can we test the spinners to check our predictions?

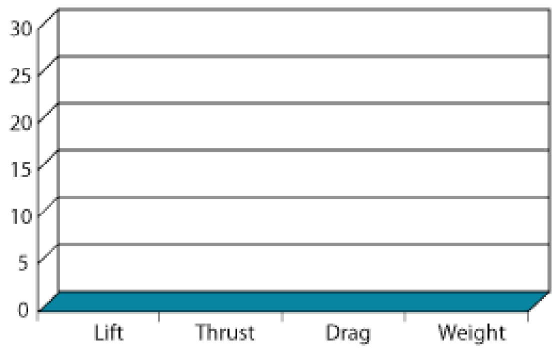
After discussing their suggestions, ask students to work with a partner to spin each spinner 50 times and to record all results in a tally table.

Spinner C Tally Table

Lift	
Thrust	
Drag	
Weight	

When all data have been collected, help students display their data in bar graphs.

Results of 50 Spins



Each bar graph should be discussed and interpreted. Help students see that on spinner C, all forces have the same chance for the pointer to land on them. Since there is 1 chance out of 4 equal chances that the pointer will land on Lift, the probability is ?.

Class Conversation

Use the following questions to guide the next discussion.

Compare the regions in spinner C. How many of the same size do you see? [4]

How many different forces are on spinner C? [4]

Is it less likely, equally likely, or more likely that the pointer will land on Lift than on Weight, Drag, or Thrust?

What is the likelihood, or probability, of landing on Thrust? On Drag? On Weight?

Does the pointer have the same chance of landing on each force? Why do you think that?

Next, ask students to look closely at their graphs for spinner C to interpret the results of their experiment.

Since the probability is the same for each force on spinner C, what should the graphs look like? Why?

Do your bar graphs show this?

Which bars are taller? Shorter?

What does the bars' appearance show you?

Repeat this analysis process with each spinner.

When it is your turn how can you use your graphs to help you decide which spinner to use on that turn?

Students should keep all spinners for the Rescue Mission Game.

Part 2: Playing the Rescue-Mission Game

Introduce or review how to read and write ordered pairs to name a location on a coordinate grid. Practice using the spinners to determine moves on the game board. Ask students to predict how many spins will be needed to reach the top of the mountain.

Students then play the game with a partner to see which helicopter can rescue the lost hikers on the mountaintop first. As they take turns, students should record the following data:

- Which spinner is selected for the turn
- Where the pointer lands (lift, thrust, weight, or drag)
- How the student moves (up, right, down, or left)
- The ordered pair that names the point to which they move

Turn Number	I Selected This Spinner	Pointer Landed On	Direction in Which I Moved	Place Where I Landed
1	C	Lift	Up	(0,1)
2	D	Thrust	Right	(1,1)
3	D	Lift	Up	(1,2)

Students can record the flight path on the game-board grid by plotting the points for each student in different colors. When finished, students can connect the points to show the flight path.

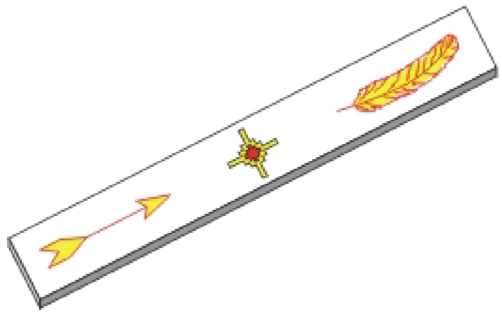
Sticks and Stones

Students will play *Sticks and Stones*, a game based on the Apache game "Throw Sticks," which was played at multi-nation celebrations. Students will collect data, investigate the likelihood of various moves, and use basic ideas of expected value to determine the average number of turns needed to win a game.

Instructional Plan

The *Sticks and Stones* game is based on the Apache game "Throw Sticks." To play the game, students throw three sticks, each decorated on one side. Students move their pieces around the game board based on the results of the throw, as described below.

Allow students to decorate three sticks on one side *only*; the other side should be blank. (If playing this game as part of a larger unit about Native American culture, you can allow students to decorate the sticks with tribal symbols.) Students will use these sticks to determine how far they move when playing the game.



To create the game board, arrange 40 stones in a circle, preferably divided into four groups of 10. (In groups of 10, a side benefit of this game is that it helps to develop student understanding of the place-value system. For instance, if a student is currently on the seventh stone in one group of 10 and rolls a 5, she gets to move to the second stone in the next group of 10. This demonstrates modular arithmetic, because 7 + 5 = 12, which has remainder 2 when divided by 10.) As an alternative, you can use a Monopoly® game board, which consists of 10 squares on each of four sides.

The rules of the game are as follows:

- **Object of the Game:** Be the first player to move your piece around the board past your starting point.
- **Set-Up:** Each student should place a marker on opposite sides of the circle. The area inside the circle is used for throwing the sticks when playing the game.

- **Play:** Determine which player will go first. Player 1 throws the three sticks into the center of the circle and moves her piece according to the results:

  Player 2 then throws the sticks and moves accordingly. Play continues with players alternating turns.

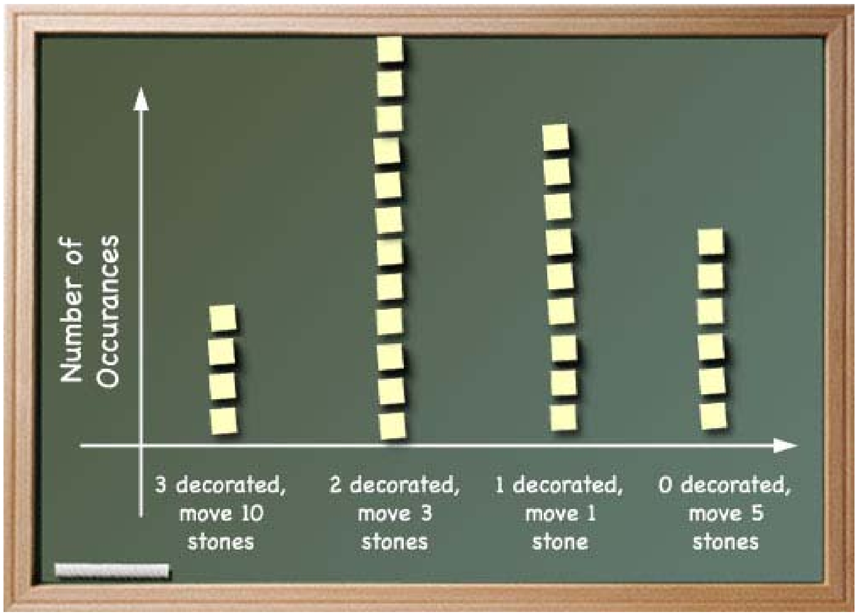
- **Special Rule:** If one player's marker lands on or passes another player's, the player passed over must move her piece back to the starting point.

Pair students together, and let them play the game once, for fun. Then, before playing a second time, have students make a chart of all throws that are possible. During a second game, have them keep track of their throws while playing. How many of each occurred?

As an alternative, students can use the demonstration below to generate random throws.

After tallying their throws during the second game, have kids use sticky notes to build a bar graph. Place a large piece of paper on the wall, or draw a graph on the chalkboard, which shows the possible throws on the horizontal axis and the number of occurrences on the vertical axis.

For each time a particular throw occurred during their games, students should place a sticky note on the graph. For instance, if a student had three throws with zero sides decorated, the student should place three sticky notes in that category. Allow 4-6 students to place sticky notes on the same graph. Compiling the data in this way will give a larger sample size and should yield experimental results that are close to the theoretical probabilities; if only 1-2 students place their data on a graph, the results are more likely to be skewed. As necessary, create a new graph for each group of 4-6 students. (If possible, you can put all of the data from the entire class on one graph, but if there is too much data, the bars will get too tall.) A completed graph may look something like the following:



Allow students to compare the relative heights of the bars on the graph. [The bars for one or two sides decorated are much taller, meaning that those results are more likely when the sticks are thrown. It also means that the probability of having a throw with three sides the same is less likely.]

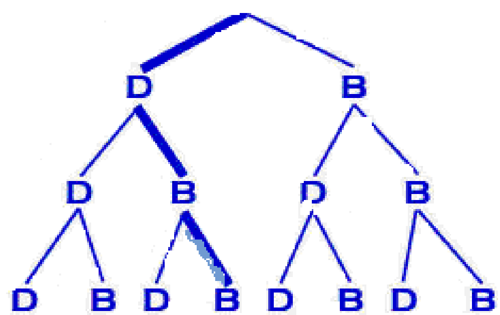
To facilitate a discussion about what the graph means, have students compare just two categories. You may want to ask the following questions:

- Which is more likely—a throw with one stick decorated or a throw with two sticks decorated? [Neither. They both occur about the same amount.]
- Which is more likely—a throw with three sticks decorated or a throw with no sticks decorated? [Neither. They both occur about the same amount.]
- Which is more likely—a throw with three sticks decorated or a throw with two sticks decorated? [A throw with two sticks decorated is about three times as likely as a throw with all three decorated.]
- Which is more likely—a throw with no sticks decorated or a throw with one stick decorated? [A throw with one stick decorated is about three times as likely as a throw with no sticks decorated.]

Be sure to use mathematical terms during this discussion, such as *likely* and *probability*. For instance, you may want to ask students, "How much more likely is it to throw three decorated sides than to throw only two decorated sides? Is it twice as likely? More than twice as likely?" [From the graph, it appears to be about three times as likely, because the bar is three times as tall.]

Return to the context of the game. Ask students, "Why do you think you get to move more spaces when all three sticks land on the same side?" [Throws with zero or three sides decorated are less likely than throws with one or two sides decorated. Since they are more rare, the reward for those throws is greater. On the other hand, a throw with three sides decorated is just as likely as a throw with no sides decorated, yet the reward for three sides decorated is greater; this is not a mathematical decision, but it probably has to do with human appreciation of art.]

The bar graph allows student to use experimental results to discuss probability, but they should also consider the theoretical probability of each result. This can be accomplished by constructing a tree diagram that shows the results after three throws; a D represents a decorated side, and a B represents a blank side:



There are eight possible outcomes, as indicated by the number of elements in the third row. The path to each of those elements indicates one possible outcome; for example, the highlighted path shows a first throw of D, a second throw of B, and a third throw of B.

An organized list could also be created. The list below shows the eight possible outcomes, which verify the results of the tree diagram:

DDD DDB DBD BDD

BBD BDB DBB BBB

Because three sticks are thrown, and because there are two possible results with each stick (D or B), it makes sense that there would be $2^3 = 8$ outcomes.

To promote conceptual understanding, be sure to compare the items on the list to the outcomes from the tree diagram. For instance, show that the highlighted path is equivalent to DBB in the list.

Based on the list and tree diagram, students should realize that three decorated sides or no decorated sides occur, on average, only once out of every eight throws, whereas one or two decorated sides occur three times every eight throws. Ask students to compare these theoretical probabilities to the experimental results they obtained when playing the game.

Finally, ask students, "On average, how many turns do you think it will take to complete a game?" Students can investigate this question by playing again and recording the number of turns, and then comparing their results with the rest of the class. Alternatively, if students are prepared for the mathematics, they can reason through the solution using basic ideas about expected value. [In eight turns, a player would be expected to get three decorated sides on one throw, two decorated sides on three throws, one decorated sides on three throws, and no decorated sides on one throw.

Consequently, the player will move $1(10) + 3(3) + 3(1) + 1(5) = 27$ stones in eight turns, or approximately $27 \div 8 = 3.375$ stones per turn. At that rate, it will take $40 \div 3.375 = 11.85$, or about 12, turns for a player to complete the circle. Of course, it will take more if the player is passed over and sent back to the starting point.]

The Game of SKUNK

In this lesson, students practice decision-making skills leading to a better understanding of choice versus chance and building the foundation of mathematical probability. This lesson is adapted from an article by Dan Brutlag, "Choice and Chance in Life: The Game of SKUNK," which appeared in *Mathematics Teaching in the Middle School*, Vol. 1, No. 1 (April 1994), pp. 28-33.

Instructional Plan

Write the following questions on the chalkboard or overhead:

- I might make more money if I was in business for myself; should I quit my job?
- An earthquake might destroy my house; should I buy insurance?
- My mathematics teacher might collect homework today; should I do it?

Ask students to share their responses to each of these scenarios. Ask students why their responses may be different from their classmates. Ideally the class discussion will mirror some of the concepts which follow.

Every day each of us must make choices like those described above. The choices we make are based on the chance that certain events might occur. We informally estimate the probabilities for events by using a variety of methods: looking at statistical information, using past experiences, asking other people's opinions, performing experiments, and using mathematical theories. Once the probability for an event has been estimated, we can examine the consequences of the event and make an informed decision about what to do.

Making the connection between choice and chance is basic to understanding the significance and usefulness of mathematical probability. We can help middle school students make this connection by giving them experiences wherein choice and change come into play followed by tasks that cause them to think about, and learn from, those experiences.

The game of SKUNK presents middle-grade students with an experience that clearly involves both choice and chance. SKUNK is a variation on a dice game also known as "pig" or "hold'em." The object of SKUNK is to accumulate points by rolling dice. Points are accumulated by making several "good" rolls in a row but choosing to stop before a "bad" roll comes and wipes out all the points. SKUNK can be played by groups, by the whole class at once, or by individuals. The whole-class version is described following an explanation of the rules.

The Game of SKUNK

To start the game each player makes a score sheet like this:

S	K	U	N	K

Each letter of SKUNK represents a different round of the game; play begins with the "S" column and continue through the "K" column. The object of SKUNK is to accumulate the greatest possible point total over five rounds. The rules for play are the same for each of the five rounds.

- At the beginning of each round, every player stands. Then, a pair of dice is rolled. (Everyone playing uses that roll of the dice; unlike other games, players do not roll the dice for just themselves.)
- A player gets the total of the dice and records it in his or her column, unless a "one" comes up.
- If a "one" comes up, play is over for that round and all the player's points in that column are wiped out.
- If "double ones" come up, all points accumulated in prior columns are wiped out as well.
- If a "one" doesn't occur, the player may choose either to try for more points on the next roll (by continuing to stand) or to stop and keep what he or she has accumulated (by sitting down).

Note: If a "one" or "double ones" occur on the very first roll of a round, then that round is over and each player must take the consequences.

Playing SKUNK with the Whole Class

The best way to teach SKUNK to the class is to play a practice game. You can use the following to simulate the rolling of number cubes by projecting the simulation onto the overhead or television screen.

Draw a SKUNK score sheet on the chalkboard or overhead transparency on which to record dice throws. Have all students make their own score sheets on their own scrap paper. Have all students stand up next to their chairs. Either you or a student rolls the dice. Suppose a "four" and a "six come up, total 10. Record the outcome of the roll in the "S" column on the chalkboard:

Score Record

S	K	U	N	K
10				

On the first roll, all the players get a total of the dice or a zero if any "ones" come up. Kerry and Lisa are standing up, so they also write "10" in their score sheets.

Kerry S 10	K	U	N	K
Lisa S 10	K	U	N	K

After each roll, players may choose either to remain standing or to sit down. Those who are standing get the results of the next dice roll; those who sit down keep the score they have accumulated for that round regardless of future dice rolls. Once someone sits down, that person may not stand up again until the beginning of the next round.

The sample game continues on the teacher sheet.

Instead of focusing on a single class winner, more students will be drawn into thinking about a strategy for doing well in this game by emphasizing personal goals. When playing the game for the second and third time, ask students to focus on trying to better their own previous score. After each game ask for a show of hands of those who did better than last time.

Thinking about SKUNK

Although playing SKUNK is fun, thinking about SKUNK is essential for student understanding of the underlying concepts. In groups of two or three, students should complete the questions on the handout.

Groups of students could organize whole-class experiments to find answers to problems 4, 5, 6. As a class, share results and solutions to the questions posed.

Suggested solutions and discussion points

For question 1, the chance part of SKUNK is the dice roll and choice part is the decision to sit down or remain standing.

Many games can be listed for question 2. Games of pure chance include Candy Land and bingo. Games involving almost pure choice, disregarding who goes first and your opponent's ability, include chess and tic-tac-toe. Most games, such as hearts, basketball, or Monopoly, involve both choice and chance. The game of Uno is mostly chance no matter what choices are made. But poker can be either mostly chance or mostly choice depending how is it played. Strategies are useful only in games that allow for choices. But even games that have choices can be mostly chance for a player who makes choices without any strategy.

Question 3 can lead to class discussions that involve interesting probabilities and decisions from students' lives. Some events that a thirteen-year-old would ascribe mostly to chance include these: you find a $20 bill, you calculator is stolen, having a bad acne outbreak, your cousin becomes a famous musician, your best friend has to go to a different high school then you, and the like. Some typical events resulting from a thirteen-year-old's choices might include these: a girl dances with you because you asked her, you flunk a quiz because you didn't study, you get paid your allowance because you do your chores, and so on.

Questions 4, 5, and 6 can be done either by experimenting or making theoretical arguments. For example, for question 5, dice could be rolled many times and the points noted. Then the points could be totaled and the average value per time calculated. One theoretical approach is to list the equally likely outcomes for rolling a pair of dice where SKUNK points are accumulated. Twenty-five equally likely outcomes yield points. Such a list of outcomes is shown in **table 1.** Rolls including a "one" are not shown because no points are accumulated on the rolls.

Table 1

		Second Die				
		2	3	4	5	6
First Die	2	4	5	6	7	8
	3	5	6	7	8	9
	4	6	7	8	9	10
	5	7	8	9	10	11
	6	8	9	10	11	12

The average of all the equally likely values is 8. This value can be either calculated or observed from the symmetry of the table.

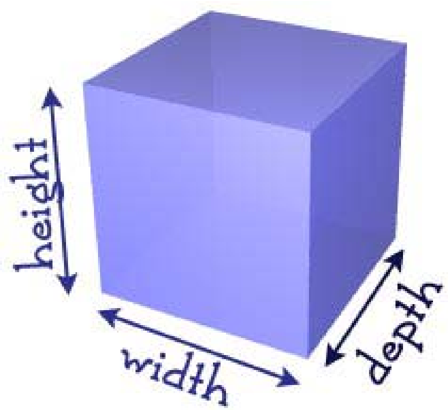
7. Solid Geometry

Three-dimensional space

Polyhedra and Non-Polyhedra

Sphere

Three Dimensions



It is called **three-dimensional**, or **3D**because there are three dimensions:*width, depth* and *height.*

Simple Shapes

Let us start with some of the simplest shapes:

- Cube
- Cuboid
- Volume of a Cuboid

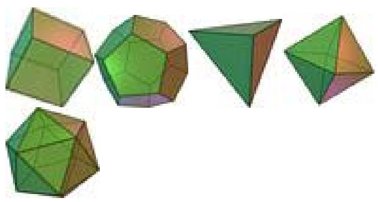
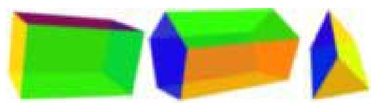
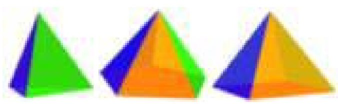
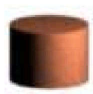
Properties

Solids have *properties* (special things about them), such as:

- volume (think of how much water it could hold)
- surface area (think of the area you would have to paint)

Polyhedra and Non-Polyhedra

There are two main types of solids, "Polyhedra", and "Non-Polyhedra":

Polyhedra : ***(they must have flat faces)***		Platonic Solids	
		Prisms	
		Pyramids	
Non-Polyhedra: ***(if any surface is not flat)***		Sphere	
		Torus	
		Cylinder	
		Cone	

Sphere

Sphere Facts

Notice these interesting things:

It is perfectly symmetrical

It has **no** edges or vertices (corners)

It is **not** a polyhedron

All points on the surface are the same distance from the center

And for reference:

Surface Area = $4 \times \boldsymbol{\pi} \times r^2$

Volume = $(4/3) \times \boldsymbol{\pi} \times r^3$

Largest Volume for Smallest Surface

Of all the shapes, a sphere has the smallest surface area for a volume. Or put another way it can contain the greatest volume for a fixed surface area.

Example: if you blow up a balloon it naturally forms a sphere because it is trying to hold as much air as possible with as small a surface as possible. Press the Play button to see.

In Nature

The sphere appears in nature whenever a surface wants to be as small as possible. Examples include bubbles and water drops, can you think of more?

The Earth

The Planet Earth, our home, is *nearly* a sphere, except that it is squashed a little at the poles.

It is a **spheroid**, which means it just misses out on being a sphere because it isn't perfect in one direction (in the Earth's case: North-South)

8. History of Matrices and determinants

The beginnings of matrices and determinants go back to the second century BC although traces can be seen back to the fourth century BC. However it was not until near the end of the 17^{th} Century that the ideas reappeared and development really got underway.

It is not surprising that the beginnings of matrices and determinants should arise through the study of systems of linear equations. The Babylonians studied problems which lead to simultaneous linear equations and some of these are preserved in clay tablets which survive. For example a tablet dating from around 300 BC contains the following problem:-

There are two fields whose total area is 1800 *square yards. One produces grain at the rate of* $^2/_3$ *of a bushel per square yard while the other produces grain at the rate of* $^1/_2$ *a bushel per square yard. If the total yield is* 1100 *bushels, what is the size of each field.*

The Chinese, between 200 BC and 100 BC, came much closer to matrices than the Babylonians. Indeed it is fair to say that the text Nine Chapters on the Mathematical Art written during the Han Dynasty gives the first known example of matrix methods. First a problem is set up which is similar to the Babylonian example given above:-

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 *measures. Two of the first, three of the second and one of the third make* 34*measures. And one of the first, two of the second and three of the third make* 26 *measures. How many measures of corn are contained of one bundle of each type?*

Now the author does something quite remarkable. He sets up the coefficients of the system of three linear equations in three unknowns as a table on a 'counting board'.

```
  1   2   3
  2   3   2
  3   1   1
 26  34  39
```

Our late 20th Century methods would have us write the linear equations as the rows of the matrix rather than the columns but of course the method is identical. Most remarkably the author, writing in 200 BC, instructs the reader to multiply the middle column by 3 and subtract the right column as many times as possible, the same is then done subtracting the right column as many times as possible from 3 times the first column. This gives

0	0	3
4	5	2
8	1	1
39	24	39

Next the left most column is multiplied by 5 and then the middle column is subtracted as many times as possible. This gives

0	0	3
0	5	2
36	1	1
99	24	39

from which the solution can be found for the third type of corn, then for the second, then the first by back substitution. This method, now known as Gaussian elimination, would not become well known until the early 19th Century.

Cardan, in *Ars Magna* (1545), gives a rule for solving a system of two linear equations which he calls *regula de modo* and which [7] calls *mother of rules* ! This rule gives what essentially is Cramer's rule for solving a 2 × 2 system although Cardan does not make the final step. Cardan therefore does not reach the definition of a determinant but, with the advantage of hindsight, we can see that his method does lead to the definition.

Many standard results of elementary matrix theory first appeared long before matrices were the object of mathematical investigation. For example de Witt in *Elements of curves*, published as a part of the commentaries on the 1660 Latin version of Descartes' *Géométrie* , showed how a transformation of the axes reduces a given equation for a conic to canonical form. This amounts to diagonalising a symmetric matrix but de Witt never thought in these terms.

The idea of a determinant appeared in Japan and Europe at almost exactly the same time although Seki in Japan certainly published first. In 1683 Seki wrote *Method of solving the dissimulated problems* which contains matrix methods written as tables in exactly the way the Chinese methods described above were constructed. Without having any word which corresponds to 'determinant' Seki still introduced determinants and gave general methods for calculating them based on examples. Using his 'determinants' Seki was able to find determinants of 2 × 2, 3 × 3, 4 × 4 and 5 × 5 matrices and applied them to solving equations but not systems of linear equations.

Rather remarkably the first appearance of a determinant in Europe appeared in exactly the same year 1683. In that year Leibniz wrote to de l'Hôpital. He explained that the system of equations

$$10 + 11x + 12y = 0$$
$$20 + 21x + 22y = 0$$
$$30 + 31x + 32y = 0$$

had a solution because

10.21.32 + 11.22.30 + 12.20.31 = 10.22.31 + 11.20.32 + 12.21.30

which is exactly the condition that the coefficient matrix has determinant 0. Notice that here Leibniz is not using numerical coefficients but

two characters, the first marking in which equation it occurs, the second marking which letter it belongs to.

Hence 21 denotes what we might write as a_{21}.

Leibniz was convinced that good mathematical notation was the key to progress so he experimented with different notation for coefficient systems. His unpublished manuscripts contain more than 50 different ways of writing coefficient systems which he worked on during a period of 50 years beginning in 1678. Only two publications (1700 and 1710) contain results on coefficient systems and these use the same notation as in his letter to de l'Hôpital mentioned above.

Leibniz used the word 'resultant' for certain combinatorial sums of terms of a determinant. He proved various results on resultants including what is essentially Cramer's rule. He also knew that a determinant could be expanded using any column - what is now called the Laplace expansion. As well as studying coefficient systems of equations which led him to determinants, Leibniz also studied coefficient systems of quadratic forms which led naturally towards matrix theory.

In the 1730's Maclaurin wrote *Treatise of algebra* although it was not published until 1748, two years after his death. It contains the first published results on determinants proving Cramer's rule for 2×2 and 3×3 systems and indicating how the 4×4 case would work. Cramer gave the general rule for $n \times n$ systems in a paper *Introduction to the analysis of algebraic curves* (1750). It arose out of a desire to find the equation of a plane curve passing through a number of given points. The rule appears in an Appendix to the paper but no proof is given:-

One finds the value of each unknown by forming n fractions of which the common denominator has as many terms as there are permutations of n things.

Cramer does go on to explain precisely how one calculates these terms as products of certain coefficients in the equations and how one determines the sign. He also says how the n numerators of the fractions can be found by replacing certain coefficients in this calculation by constant terms of the system.

Work on determinants now began to appear regularly. In 1764 Bezout gave methods of calculating determinants as did Vandermonde in 1771. In 1772 Laplace claimed that the methods introduced by Cramer and Bezout were impractical and, in a paper where he studied the orbits of the inner planets, he discussed the solution of systems of linear equations without actually calculating it, by using determinants. Rather surprisingly Laplace used the word 'resultant' for what we now call the determinant: surprising since it is the same word as used by Leibniz yet Laplace must have been unaware of Leibniz's work. Laplace gave the expansion of a determinant which is now named after him.

Lagrange, in a paper of 1773, studied identities for 3 × 3 functional determinants. However this comment is made with hindsight since Lagrange himself saw no connection between his work and that of Laplace and Vandermonde. This 1773 paper on mechanics, however, contains what we now think of as the volume interpretation of a determinant for the first time. Lagrange showed that the tetrahedron formed by O(0,0,0) and the three points $M(x,y,z)$, $M'(x',y',z')$, $M''(x'',y'',z'')$ has volume

$$\frac{1}{6}\,[z(x'y'' - y'x'') + z'(yx'' - xy'') + z''(xy' - yx')].$$

The term 'determinant' was first introduced by Gauss in *Disquisitiones rithmeticae* (1801) while discussing quadratic forms. He used the term because the determinant determines the properties of the quadratic form. However the concept is not the same as that of our determinant. In the same work Gauss lays out the coefficients of his quadratic forms in rectangular arrays. He describes matrix multiplication (which he thinks of as composition so he has not yet reached the concept of matrix algebra) and the inverse of a matrix in the particular context of the arrays of coefficients of quadratic forms.

Gaussian elimination, which first appeared in the text Nine Chapters on the Mathematical Art written in 200 BC, was used by Gauss in his work which studied the orbit of the asteroid Pallas. Using observations of Pallas taken between 1803 and 1809, Gauss obtained a system of six linear equations in six unknowns. Gauss gave a systematic method for solving such equations which is precisely Gaussian elimination on the coefficient matrix.

It was Cauchy in 1812 who used 'determinant' in its modern sense. Cauchy's work is the most complete of the early works on determinants. He reproved the earlier results and gave new results of his own on minors and adjoints. In the 1812 paper the multiplication theorem for determinants is proved for the first time although, at the same meeting of the Institut de

France, Binet also read a paper which contained a proof of the multiplication theorem but it was less satisfactory than that given by Cauchy.

In 1826 Cauchy, in the context of quadratic forms in n variables, used the term 'tableau' for the matrix of coefficients. He found the eigenvalues and gave results on diagonalisation of a matrix in the context of converting a form to the sum of squares. Cauchy also introduced the idea of similar matrices (but not the term) and showed that if two matrices are similar they have the same characteristic equation. He also, again in the context of quadratic forms, proved that every real symmetric matrix is diagonalisable.

Jacques Sturm gave a generalisation of the eigenvalue problem in the context of solving systems of ordinary differential equations. In fact the concept of an eigenvalue appeared 80 years earlier, again in work on systems of linear differential equations, by D'Alembert studying the motion of a string with masses attached to it at various points.

It should be stressed that neither Cauchy nor Jacques Sturm realised the generality of the ideas they were introducing and saw them only in the specific contexts in which they were working. Jacobi from around 1830 and then Kronecker and Weierstrass in the 1850's and 1860's also looked at matrix results but again in a special context, this time the notion of a linear transformation. Jacobi published three treatises on determinants in 1841. These were important in that for the first time the definition of the determinant was made in an algorithmic way and the entries in the determinant were not specified so his results applied equally well to cases were the entries were numbers or to where they were functions. These three papers by Jacobi made the idea of a determinant widely known.

Cayley, also writing in 1841, published the first English contribution to the theory of determinants. In this paper he used two vertical lines on either side of the array to denote the determinant, a notation which has now become standard.

Eisenstein in 1844 denoted linear substitutions by a single letter and showed how to add and multiply them like ordinary numbers except for the lack of commutativity. It is fair to say that Eisenstein was the first to think of linear substitutions as forming an algebra as can be seen in this quote from his 1844 paper:-

An algorithm for calculation can be based on this, it consists of applying the usual rules for the operations of multiplication, division, and exponentiation to symbolic equations between linear systems, correct symbolic equations are always obtained, the sole consideration being that the order of the factors may not be altered.

The first to use the term 'matrix' was Sylvester in 1850. Sylvester defined a matrix to be *an oblong arrangement of terms* and saw it as something which led to various determinants from square arrays contained within it. After leaving America and returning to England in 1851, Sylvester became a lawyer and met Cayley, a fellow lawyer who shared his interest in mathematics. Cayley quickly saw the significance of the matrix concept and by 1853 Cayley had published a note giving, for the first time, the inverse of a matrix.

Cayley in 1858 published *Memoir on the theory of matrices* which is remarkable for containing the first abstract definition of a matrix. He shows that the coefficient arrays studied earlier for quadratic forms and for linear transformations are special cases of his general concept. Cayley gave a matrix algebra defining addition, multiplication, scalar multiplication and inverses. He gave an explicit construction of the inverse of a matrix in terms of the determinant of the matrix. Cayley also proved that, in the case of 2×2 matrices, that a matrix satisfies its own characteristic equation. He stated that he had checked the result for 3×3 matrices, indicating its proof, but says:-

I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a matrix of any degree.

That a matrix satisfies its own characteristic equation is called the Cayley-Hamilton theorem so its reasonable to ask what it has to do with Hamilton. In fact he also proved a special case of the theorem, the 4×4 case, in the course of his investigations into quaternions.

In 1870 the Jordan canonical form appeared in *Treatise on substitutions and algebraic equations* by Jordan. It appears in the context of a canonical form for linear substitutions over the finite field of order a prime.

Frobenius, in 1878, wrote an important work on matrices *On linear substitutions and bilinear forms* although he seemed unaware of Cayley's work. Frobenius in this paper deals with coefficients of forms and does not use the term matrix. However he proved important results on canonical matrices as representatives of equivalence classes of matrices. He cites Kronecker and Weierstrass as having considered special cases of his results in 1874 and 1868 respectively. Frobenius also proved the general result that a matrix satisfies its characteristic equation. This 1878 paper by Frobenius also contains the definition of the rank of a matrix which he used in his work on canonical forms and the definition of orthogonal matrices.

The nullity of a square matrix was defined by Sylvester in 1884. He defined the nullity of A, $n(A)$, to be the largest i such that every minor of A of order $n-i+1$ is zero. Sylvester was interested in invariants of matrices, that is properties which are not changed by certain transformations. Sylvester proved that

$$\max\{n(A), n(B)\} \le n(AB) \le n(A) + n(B).$$

In 1896 Frobenius became aware of Cayley's 1858 *Memoir on the theory of matrices* and after this started to use the term matrix. Despite the fact that Cayley only proved the Cayley-

Hamilton theorem for 2×2 and 3×3 matrices, Frobenius generously attributed the result to Cayley despite the fact that Frobenius had been the first to prove the general theorem.

An axiomatic definition of a determinant was used by Weierstrass in his lectures and, after his death, it was published in 1903 in the note *On determinant theory*. In the same year Kronecker's lectures on determinants were also published, again after his death. With these two publications the modern theory of determinants was in place but matrix theory took slightly longer to become a fully accepted theory. An important early text which brought matrices into their proper place within mathematics was *Introduction to higher algebra* by Bôcher in 1907. Turnbull and Aitken wrote influential texts in the 1930's and Mirsky's *An introduction to linear algebra* in 1955 saw matrix theory reach its present major role in as one of the most important undergraduate mathematics topic.

Article by: *J J O'Connor* and *E F Robertson*

9. The Language of Mathematics

Mathematical results are expressed in a foreign language.

It is worth quoting at some length from Devlin's conclusion:

A mathematical study of any one phenomenon has many similarities to a mathematical study of any other. There is an initial simplification, in which the key concepts are identified and isolated. Then those key concepts are analysed in greater and greater depth, as relevant patterns are discovered and investigated. There are attempts at axiomatisation. The level of abstraction increases. Theorems are formulated and proved. Connections to other parts of mathematics are uncovered or suspected. The theory is generalised, leading to the discovery of further similarities to - and connections with - other areas of mathematics."

The Language of Mathematics

The Language of Mathematics

The Language of Mathematics was designed so we can write about:

Things like Numbers, Sets, Functions, etc

What we Do with those things (add, subtract, multiply, divide, join together, etc)

Symbols

Mathematics uses symbols instead of words:

- There are the 10 digits: 0,1,2,...9
- There are symbols for operations: + - x /
- And symbols that "stand in" for values: x, y, ...
- And many special symbols: = < ≤, ...

Letter Conventions

Often (but not always) letters have special uses:

	Examples	What they usually mean
Start of the alphabet:	a, b, c, ...	constants (fixed values)
From i to n:	i, j, k, ..., n	positive integers (for counting)

End of the alphabet: ... x, y, z variables (unknowns)

Those are **not rules**, but they are often used that way.

Example:

$y = ax + b$

People would **assume** that a and b are fixed values,

And that x is the one that changes, which in turn makes y change.

Nouns, Verbs, Sentences

Even though we don't actually use the words "noun", "verb", or "pronoun" in Mathematics, it might help you understand Mathematics by thinking about its **similarities to English**:

Nouns

Nouns could be fixed things, such as numbers, or expressions with numbers

15	2(3-1/2)	4^2

Verbs

And the verb could be the equals sign "=", or maybe an inequality like < or >

Pronouns

in English pronouns are things that stand in for nouns (it, he, you, etc). In Mathematics they could be variables like **x** or **y**:

5**x**-7	$\mathbf{xy}^2$	-3/**x**

Sentence

And they could be put together into a sentence like this:

$$3x + 7 = 22$$

(And we actually do use the word sentence in mathematics!)

That is why, like other languages, the language of mathematics has its own grammar, syntax, vocabulary, word order, synonyms, negations, conventions, abbreviations, sentence structure, and paragraph structure. It has certain language features unparalleled in other languages (for example, theorems expressed using the letter "x" also apply to "b" and "$2x$-5").

To teach essential language concepts which have been underemphasized in the usual mathematics curriculum. To emphasize the basic patterns of mathematical expression and thought. There are a limited number of frequently repeated patterns of expression and thought in Mathematics. This text identifies, isolates, and emphasizes the essential patterns, illustrating them in several subject areas of mathematics.

There are a limited number of key vocabulary words from logic ("and", "or", "not", "if... then", "if and only if", "for all", and "there exists") which are frequently used in mathematics.

Students will learn to read math. The text teaches how to read math well enough in order to learn math by reading. It sounds like a tall order, but it works!

What is **different** about *The Language of Mathematics*?

- **Everything!**
- Constant emphasis of **patterns** of thought and expression which recur throughout mathematics
- Thorough explanation of what makes mathematics "algebra" and how to think "in algebra."
- Emphasis on bringing the students **up** to a mathematical, abstract, level of expression and understanding
- Emphasis on mathematical examples of sentences and reasoning (not logic of this sort: "If it's raining, then I will get wet...")
- Emphasis on alternative ways to express the same information until students are comfortable with all the ways mathematical thoughts are expressed
    - logical equivalences
    - letter-switching
    - theorems which use "iff"
    - definitions
    - English v. mathematical expression
    - abbreviations, notation
    - Making implicit usages explicit
- Little equation-solving until they have the ability to read the theorems which justify the steps (learning to **read** in order to learn is a major thrust of the text). This is not a calculation-oriented text.
- Algebraic methods are justified (and students understand the justifications)

- Proofs are introduced near the end, after students have all the background they need.

Index for Geometry

Math terminology from plane and solid geometry. This includes basic triangle trigonometry as well as a few facts not traditionally taught in basic geometry.

Bisect Bisector Box Cartesian Coordinates Cartesian Form Cartesian Plane Cavalieri’s Principle Center of Rotation Centers of a Triangle Central Angle Centroid Centroid Formula Ceva’s Theorem Cevian Chord Circle Circular Cone Circular Cylinder Circular Functions Circumcenter	Circumcircle Circumference Circumscribable Circumscribed Circumscribed Circle Coincident Collinear Complement of an Angle Complementary Angles Composite Compression Compression of a Geometric Figure Compute Concave Concentric Concurrent Cone Cone Angle Congruence Tests for Triangles Congruent	Consecutive Interior Angles Contrapositive Contraction Convex Coordinate Geometry Coordinate Plane Coordinates Coplanar Corresponding cos $\cos^{-1}$ Cos^{-1} cosine CPCFC CPCTC Cube Cube Root Cuboid Cylinder Decagon

Degenerate Degree (angle measure) Diagonal of a Polygon Diameter	Equation of a Line Equiangular Triangle Equidistant Equilateral Triangle	Geometry Glide Glide Reflection Golden Mean

Diametrically Opposed Dihedral Angle Dilation Dilation of a Geometric Figure Dimensions Direct Proportion Direct Variation Directly Proportional Disk Distance Formula Distinct Dodecagon Dodecahedron Double Cone Edge of a Polyhedron Elliptic Geometry	Euclidean Geometry Euler Line Euler's Formula (Polyhedra) Evaluate Exact Values of Trig Functions Exterior Angle of a Polygon Face of a Polyhedron Fibonacci Sequence Fixed Flip Formula Fractal Frustum of a Cone or Pyramid Geometric Figure Geometric Mean Geometric Solid	Golden Ratio Golden Rectangle Golden Spiral Graph of an Equation or Inequality Great Circle Height Height of a Cone Height of a Cylinder Height of a Parallelogram Height of a Prism Height of a Pyramid Height of a Trapezoid Height of a Triangle Heptagon Hero’s Formula Heron’s Formula

Hexagon Hexahedron HL Congruence HL Similarity Horizontal Hyperbolic Geometry Hypotenuse Icosahedron	Isometry Isosceles Trapezoid Isosceles Triangle Kite Lateral Area Lateral Surface Area Lateral Surface/Face Law of Cosines	Measure of an Angle Measurement Median of a Trapezoid Median of a Triangle Menelaus’s Theorem Mensuration Midpoint Midpoint Formula

Image of a Transformation	Law of Sines	Minor Arc
Incenter	Leg of an Isosceles Triangle	Minute
Incircle	Leg of a Right Triangle	Möbius Strip
Inradius	Leg of a Trapezoid	Negative Reciprocal
Inscribed Angle in a Circle	Line	n-gon
Inscribed Circle	Line Segment	No Slope
Interior	Linear	Non-Adjacent
Interior Angle	Linear Pair of Angles	Nonagon
Invariant	Locus	Noncollinear
Inverse Cosine	Magnitude	Non-Convex
Inverse Sine	Major Arc	Non-Euclidean Geometry
Inverse Tangent	Mean	Number Line

Oblique	Parallel Planes	Pre-Image of a Transformation
Oblique Cone	Parallel Postulate	Prism
Oblique Cylinder	Parallelepiped	Proportional
Oblique Prism	Parallelogram	Pyramid
Oblique Pyramid	Pentagon	Pythagorean Identities
Obtuse Angle	Perimeter	Pythagorean Theorem
Obtuse Triangle	Perpendicular	Pythagorean Triple
Octagon	Perpendicular Bisector	Quadrangle
Octahedron	Phi (Φ φ)	Quadrants
Octants	Pi (Π π)	Quadrilateral
One Dimension	Plane	Radian
Opposite Reciprocal	Plane Figure	Radical
Ordered Pair	Plane Geometry	

Ordered Triple	Platonic Solids	Radicand
Ordinate	Point	Radius of a Circle or Sphere
Origin	Point of Symmetry	Ray
Orthocenter	Polygon	Rectangle
Oval	Polygon Interior	Rectangular Coordinates
Pappus's Theorem	Polyhedron	Rectangular Parallelepiped
Parallel Lines	Precision	Reflection
		Regular Dodecahedron

Regular Hexahedron	Right Regular Pyramid	Side of a Polygon
Regular Icosahedron	Right Square Prism	Similar
Regular Octahedron	Right Triangle	Similarity Tests for Triangles
Regular Polygon	Root of a Number	sin
Regular Polyhedra	Rotation	$\sin^{-1}$
Regular Prism	SAA Congruence	Sin^{-1}
Regular Pyramid	SAS Congruence	Sine
Regular Right Prism	SAS Similarity	Skew Lines
Regular Right Pyramid	Scale Factor	Slant Height
Regular Tetrahedron	Scalene Triangle	Slope of a Line
Rhombus	Secant Line	SOHCAHTOA
Riemannian Geometry	Second	Solid
Right Angle	Sector of a Circle	Solid Geometry
Right Circular Cone	Segment	Sphere
Right Circular Cylinder	Segment of a Circle	Square
Right Cone	Self-Similarity	Square Root
Right Cylinder	Semicircle	

Right Prism Right Pyramid Right Regular Prism	Semiperimeter Shift Side of an Angle	SSA SSS Congruence SSS Similarity Standard Position

Stewart's Theorem Straight Angle Supplement Supplementary Angles Surd Surface Surface Area Symmetric Takeout Angle tan $\tan^{-1}$ Tan^{-1} Tangent (Trig Function) Tangent Line Tau (Τ τ) Tessellate Tetrahedron Theorem of Menelaus Theorem of Pappus	Theta (Θ θ) Three Dimensional Coordinates Three Dimensions Torus Transformations Translation Transversal Trapezium Trapezoid Triangle Triangle Congruence Tests Triangle Inequality Triangle Similarity Tests Trig Trig Functions Trig Values of Special Angles Trigonometry Truncated Cone or Pyramid Truncated Cylinder or Prism	Two Dimensions Undecagon Undefined Slope Unit Circle Varignon Parallelogram of a Quadrilateral Vertex Vertical Vertical Angles Volume Washer x-intercept x-y Plane x-z Plane y-intercept y-z Plane z-intercept Zero Dimensions Zero Slope

Terminology of Units

Name	Examples	Explanation
Derived unit	metres per second square centimetres litres per hour cubic kilometres metres per second per second watt	A derived unit measures a quantity made from a combination of other quantities. For example, an object has a speed of 1 km per hour (derived unit) if it travels 1 kilometre in one hour. Many everyday and scientific quantities require derived units, eg velocity and speed (distance per unit time); acceleration (velocity change per unit time); force (mass unit times acceleration unit); interest rate (money per year).
SI unit or metric unit	metre kilometre metres per second square metre	Any unit in the "metric system" . These are all base-ten compatible (except where they involve time). SI stands for 'international system' in French (Système Internationale d'Unités)
Standard unit	metre (m) kilogram (kg) second (s)	There are 7 units which are used to make the agreed SI units for all known physical quantities. For example, the newton, the unit for force is defined in terms of three of the standard units: kilograms, metres and seconds. There are only three standard units which are common in school mathematics - kg, m, s. Other standard units are for electric current (ampere), temperature (degrees kelvin), amount of substance (mole)and light intensity (candela).
Formal unit	hour degree Celsius degree (for angles) metre foot	Any unit with an agreed definition across society. They include metric and imperial units.
Informal unit	the length of my foot the mass of a Lego block time for a handclap	A term used by teachers for impermanent units used to teach students the principles of measurement.
Imperial unit	foot ounce mile inch hour	Any unit from the old British system (used in Australia before 1980s).

Abbreviations

quantity to measure	SI unit	approved abbreviation
length	metre	m
mass	kilogram	kg
time	second	s

NOTE: Other abbreviations which are strictly not correct are used occasionally. Teachers should adhere to the standard abbreviations but accept other common forms.

Common prefixes for SI units

We can combine these basic units with prefixes to form a multiple unit of more convenient size. See the table below for some of these commonly used prefixes.

prefix and symbol	value and meaning	example
mega (M)	1 000 000 one million	a megalitre is one million litres
kilo (k)	1 000 one thousand	a kilogram is one thousand grams
deci (d)	0.1 one tenth	a decimetre is one tenth of a metre
centi (c)	0.01 one hundredth	a centimetre is one hundredth of a metre
milli (m)	0.001 one thousandth	a milligram is one thousandth of a gram
micro (µ)	0.000 0001 one millionth	a micrometre is one millionth of a metre. µ is a Greek letter, pronounced as 'mu'.

Non SI-Units

Some units not in the SI system have been retained because of their practical importance. In Australia, there is common usage of the following non-SI units:

Unit	**Definition**	**Value**
minute	1 min = 60 s	60 s
hour	1 hour = 3600 s	3600 s
temperature	1 degree Celsius = 1 kelvin	1° C

Strictly speaking, 'tonne, 'litre' and 'hectare' are metric but not SI units. (The alternative SI terms, which are not commonly used, are, Mkg, 1 dm^3 and 10000 $m^{2,}$ respectively).

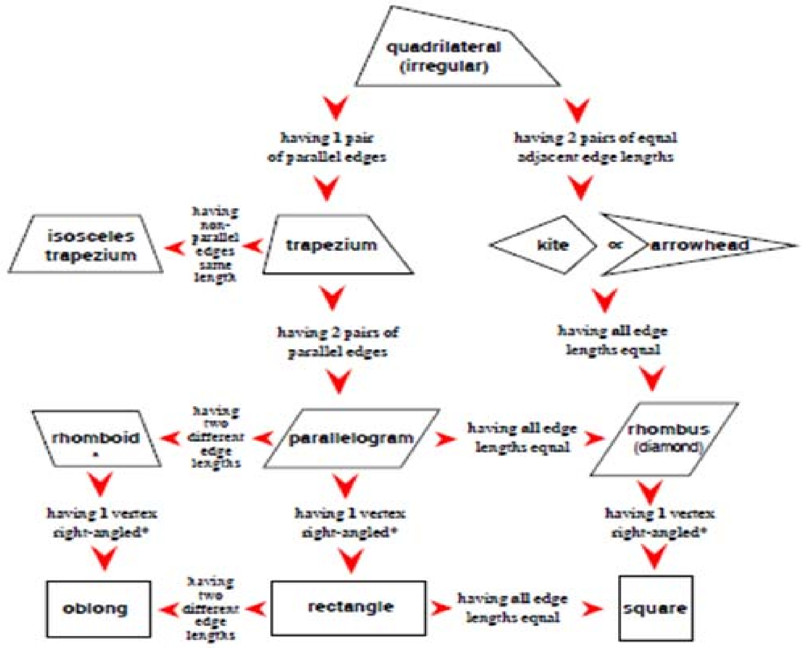
Key word summary:

Addition	**Subtraction**	**Multiplication**	**Division**
sum	less than	product	divide evenly
plus	more than	of	cut
and	decrease	multiplied	split
total	difference	times	each
increase	reduce	as much	every
more	change	by	average
raise	lost	twice	equal pieces
both	nearer		out of
combined	farther		ratio
in all	left		shared
altogether	remain		quotient
additional	fell		
extra	dropped		

The classification of quadrilaterals is a mathematical topic that is often written about. Usually the writers show different ways of doing it , illustrating their system by using either a ‘tree’ approach or a Venn diagram . Most seem to conclude that any system is ‘unsatisfactory’ in some way or other and, often, produce a new name for a shape in order to regularize the system. This might give some idea of why pupils can have difficulties at times in trying to understand what we are driving at. Looking at the definitions in nearly all major dictionaries and we see that:

a square is a rhombus is a parallelogram is a trapezium is a quadrilateral (is a polygon)

This map is one way of showing the development.



10. Introduction to Inequalities

Inequality tells you about the **relative size** of two values.

Mathematics is not always about "equals"! Sometimes you only know that something is bigger or smaller

Example: Alex and Billy have a race, and Billy wins!

What do we know?

We don't know **how fast** they ran, but we do know that Billy was faster than Alex:

Billy was faster than Alex

We can write that down like this:

$$b > a$$

(Where "b" means how fast Billy was, ">" means "greater than", and "a" means how fast Alex was)

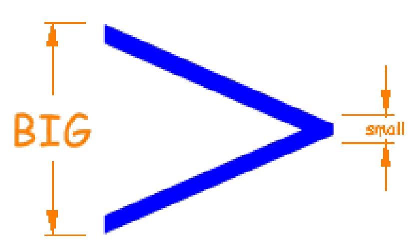
We call things like that **inequalities** (because they are not "equal")

Greater or Less Than

The two most common inequalities are:

Symbol	Words	Example Use
>	greater than	5 > 2
<	less than	7 < 9

They are easy to remember: the "small" end always points to the smaller number, like this:



Greater Than Symbol: **BIG > small**

Example: Alex plays in the under 15s soccer. How old is Alex?

We don't know **exactly** how old Alex is, because it doesn't say "equals"

But we **do know** "less than 15", so we can write:

Age < 15

The small end points to "Age" because the age is smaller than 15.

... Or Equal To!

You can also have inequalities that include "equals", like:

Symbol	Words	Example Use
≥	greater than **or equal to**	x ≥ 1
≤	less than **or equal to**	y ≤ 3

Example: you must be 13 or older to watch a movie.

The "inequality" is between **your age** and the **age of 13**.

Your age must be "greater than **or** equal to 13", which would be written:

Age ≥ 13

Comparing Values

Solving Inequalities

Sometimes we need to solve Inequalities like these:

Symbol	Words	Example
>	greater than	x + 3 > 2
<	less than	7x < 28
≥	greater than or equal to	5 ≥ x - 1
≤	less than or equal to	2y + 1 ≤ 7

Solving

Our aim is to have x (or whatever the variable is) **on its own** on the left of the inequality sign:

Something like: $x < 5$

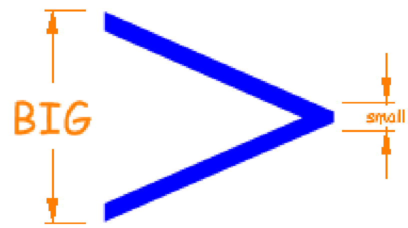
or: $y \geq 11$

We call that "solved".

How to Solve

Solving inequalities is very like solving equations ... you do most of the same things ...

... but you must also pay attention to the **direction of the inequality**.



Direction: Which way the arrow "points"

Some things you do will **change the direction**!

$<$ would become $>$

$>$ would become $<$

$\leq$ would become $\geq$

$\geq$ would become $\leq$

Safe Things To Do

These are things you can do **without affecting** the direction of the inequality:

- Add (or subtract) a number from both sides
- Multiply (or divide) both sides by a **positive** number
- Simplify a side

Example: $3x < 7+3$

You can simplify 7+3 without affecting the inequality:

$$3x < 10$$

But these things will change the direction of the inequality ("<" becomes ">" for example):

- Multiply (or divide) both sides by a **negative** number
- Swapping left and right hand sides

Example: 2y+7 < 12

When you swap the left and right hand sides, you must also **change the direction of the inequality**:

$$12 > 2y+7$$

Here are the details:

Adding or Subtracting a Value

We can often solve inequalities by adding (or subtracting) a number from both sides (just as in Introduction to Algebra), like this:

Solve: x + 3 < 7

If we subtract 3 from both sides, we get:

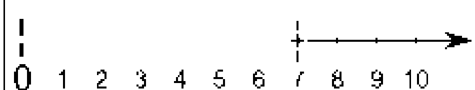
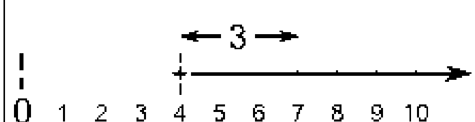
$$x + 3 \mathbf{- 3} < 7 \mathbf{- 3}$$

$$x < 4$$

And that is our solution: **x < 4**

In other words, **x** can be any value less than 4.

What did we do?

We went from this:				x+3 < 7
To this:				x < 4

And that works well for **adding** and **subtracting**, because if you add (or subtract) the same amount from both sides, it does not affect the inequality

Example: Alex has more coins than Billy. If both Alex and Billy get three more coins each, Alex will still have more coins than Billy.

What If I Solve It, But "x" Is On The Right?

No matter, just swap sides, but **reverse the sign** so it still "points at" the correct value!

Example: **12 < x + 5**

If we subtract 5 from both sides, we get:

$$12 \mathbf{- 5} < x + 5 \mathbf{- 5}$$

$$7 < x$$

That is a solution!

But it is normal to put "x" on the left hand side ...

... so let us flip sides (and the inequality sign!):

$$x > 7$$

Do you see how the inequality sign still "points at" the smaller value (7) ?

And that is our solution: **x > 7**

Note: "x" **can** be on the right, but people usually like to see it on the left hand side.

Multiplying or Dividing by a Value

Another thing we do is multiply or divide both sides by a value (just as in Algebra - Multiplying).

But we need to be a bit more careful (as you will see).

Positive Values

Everything is fine if you want to multiply or divide by a **positive number**:

Solve: 3y < 15

If we divide both sides by 3 we get:

$$3y/\mathbf{3} < 15/\mathbf{3}$$

$$y < 5$$

And that is our solution: **y < 5**

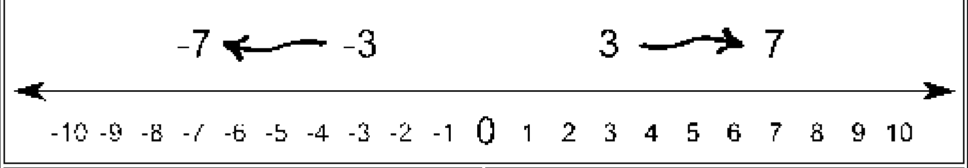
Negative Values

When you multiply or divide by a **negative number**
you have to **reverse** the inequality.

Why?

Well, just look at the number line!

For example, from 3 to 7 is **an increase**,
but from -3 to -7 is **a decrease.**



$-7 < -3$	$7 > 3$

See how the inequality sign reverses (from < to >) ?

Let us try an example:

Solve: -2y <-8

Let us divide both sides by -2 ... and **reverse the inequality**!

$$-2y < -8$$

$$-2y/\mathbf{-2} > -8/\mathbf{-2}$$

$$y > 4$$

And that is the correct solution: **y > 4**

(Note that I reversed the inequality **on the same line** I divided by the negative number.)

So, just remember:

When multiplying or dividing by a negative number, **reverse** the inequality

Multiplying or Dividing by Variables

Here is another (tricky!) example:

Solve: bx < 3b

It seems easy just to divide both sides by **b**, which would give us:

x < 3

... but wait ... if **b** is **negative** we need to reverse the inequality like this:

x > 3

But we don't know if b is positive or negative, so **we can't answer this one**!

To help you understand, imagine replacing **b** with **1** or **-1** in that example:

- if **b is 1**, then the answer is simply **x < 3**
- but if **b is -1**, then you would be solving **-x < -3**, and the answer would be **x > 3**

So:

Do not try dividing by a variable to solve an inequality (unless you know the variable is always positive, or always negative).

A Bigger Example

Solve: (x-3)/2 < -5

First, let us clear out the "/2" by multiplying both sides by 2.

Because you are multiplying by a positive number, the inequalities will not change.

$$(x-3)/2 \times \mathbf{2} < -5 \times \mathbf{2}$$

$$(x-3) < -10$$

Now add 3 to both sides:

$$x-3 + \mathbf{3} < -10 + \mathbf{3}$$

$$x < -7$$

And that is our solution: **x < -7**

Two Inequalities At Once!

How could you solve something where there are two inequalities at once?

Solve:

$$-2 < (6-2x)/3 < 4$$

First, let us clear out the "/3" by multiplying each part by 3:

Because you are multiplying by a positive number, the inequalities will not change.

$$-6 < 6-2x < 12$$

Now subtract 6 from each part:

$$-12 < -2x < 6$$

Now multiply each part by -(1/2).

Because you are multiplying by a **negative** number, the inequalities **change direction**.

$$6 > x > -3$$

And that is the solution!

But to be neat it is better to have the smaller number on the left, larger on the right. So let us swap them over (and make sure the inequalities point correctly):

$$-3 < x < 6$$

Summary

- Many simple inequalities can be solved by adding, subtracting, multiplying or dividing both sides until you are left with the variable on its own.
- But these things will change direction of the inequality:
- Multiplying or dividing both sides by a negative number
- Swapping left and right hand sides
- Don't multiply or divide by a variable (unless you know it is always positive or always negative)

11. **Co-ordinate Geometry – Distance and Midpoint** (Teacher's note)

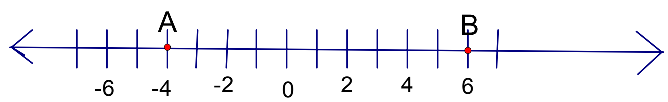
Question:

How do you determine the <u>distance</u> between two points on a coordinate plane?

How do you determine the <u>midpoint</u> between two points on a coordinate plane?

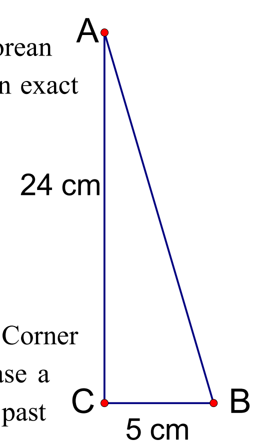
Launch:

1. Find the distance between A and B on the number line below. Explain at least two ways you could find this distance. Try to find more than two. In counting, watch for students who count tick marks instead of spaces. $|\boldsymbol{A}|+|\boldsymbol{B}|$ **or** $|\mathbf{B\text{-}A}|$



2. Find the midpoint of $\overline{AB}$ on the number line above. Explain at least two ways you could find the midpoint. Again emphasize at least two ways. Make connection with average.

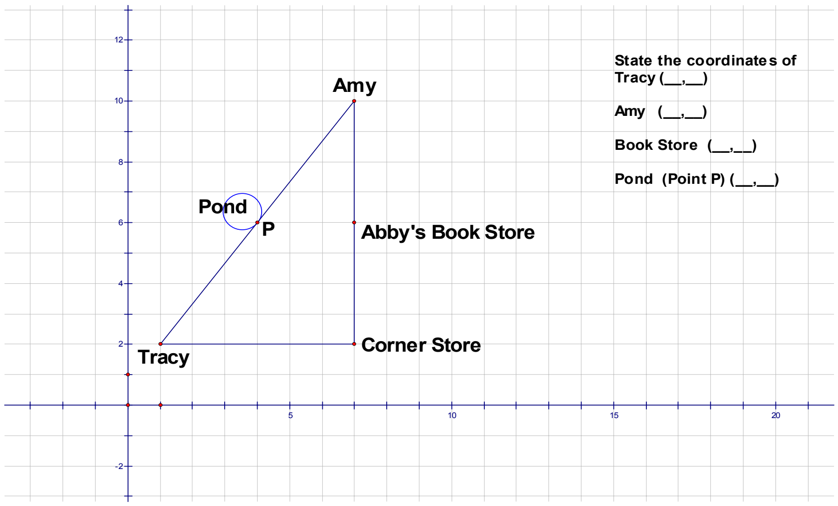
3. Find the length of $\overline{AB}$ in the right triangle above. Pythagorean Theorem is covered in middle school. Discuss difference between exact answer and approximation.



Investigation:

Tracy wants to visit Amy for her birthday. She decides to walk to the Corner Store and then pass Abby's Book Store on the way in order to purchase a present. Coming home she will take the shortcut through the park and past the pond.

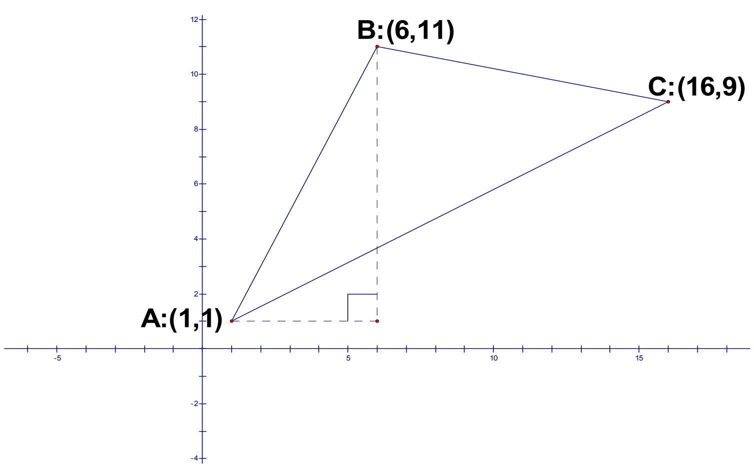
1. If each unit on the grid represents one block, how many blocks will she walk going to Amy's? How many blocks will she walk going home?



2. Explain how you computed the distance for both trips – coming and going. Give at least two ways of computing the distances.

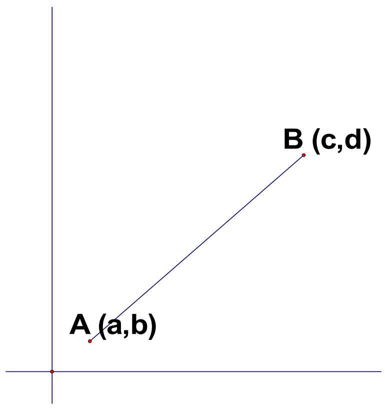
3. As Tracy is walking home through the field, she stops to dangle her feet in the pond that is exactly half way between Amy and Tracy's house. Give the coordinates of the pond. Explain how you found these coordinates. Multiple explanations.

4. A surveyor must determine the distance around a triangle formed by three towns. He put a grid over the map to help him determine these measurements as shown below. The axes are labeled in miles. Can you help the surveyor find the distance between each town? Maybe the dotted lines that form a right triangle will help you. Explain how you found these distances.Teacher could encourage them to draw more right triangles with dotted lines.



5. A shopping center is to be built midway between Towns A and C. Use the coordinates of A and C to find the coordinates for the shopping center. Explain how you found the mid-point.

6. Generalize how to find the distance between A and B below? Students should use the Pythagorean Theorem to develop the Distance Formula. May need to ask some leading questions but encourage students to develop this on their own.



Look back at how you found the coordinates of the pond in the first problem. Can you generalize this procedure and find the mid-point of segment $\overline{AB}$ above? Students should end with a formula for mid-point and distance.

Conclusions:

Write a formula (or procedure) to determine the distance between points (x_1, y_1) and (x_2, y_2). Students should be able to use their own procedure as long as they can explain it carefully.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Write a formula to determine the midpoint of the segment with endpoints (x_1, y_1) and (x_2, y_2). Students should be able to use their own procedure as long as they can explain it carefully. $M(x, y) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

In Class Problems:

1. Find the distance between (-8, 7) and (12, -9).

2. Find the midpoint of the segment with endpoints (-8, 7) and (12, -9).

3. Three vertices of a triangle are X(2, 2), Y(5, 6) and Z(7, 2). Is the triangle equilateral, scalene, or isosceles? Students may need a quick reminder here.

4. The point (9, -4) is the midpoint of a segment. One endpoint of the segment is located at (-8, 2). Find the other endpoint.

Closure: (Explain in full sentences.)

How do you determine the <u>distance</u> between two points on a coordinate plane?

How do you determine the <u>midpoint</u> between two points on a coordinate plane?

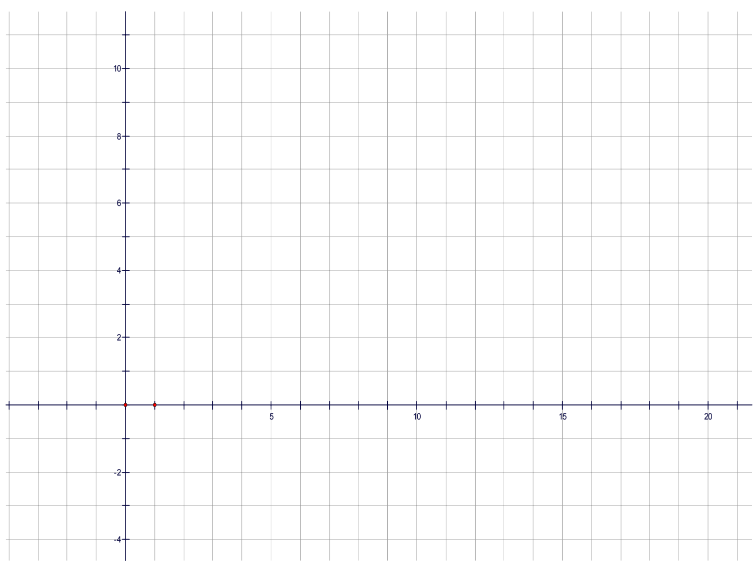
Homework:

1. Find the distance between the given points: Teacher could add one or two involving fractions or decimals.

a. (6,20) and (0, -8)

b. (-1, -1) and (-9, -11)

2. A surveyor locates the water well at (2,-3). The farmer need to pipe this water to the cattle pond at (10.4,11.2). Sketch these two locations and find the length of the pipe. Assume the grid is marked in miles.



3. Find the midpoint of the segments with the given endpoints:

a. (4, -12) and (-6, 4)

b. (18, 28) and (-7, 15)

4. Find the other endpoint of a line segment with the given midpoint and endpoint:

a. midpoint (6, -1); endpoint (-9,0)

b. midpoint (-8, -8); endpoint (12, 13)

5. How is the distance formula related to the Pythagorean Theorem?

6. Prove/verify that the midpoint of the hypotenuse of a right triangle is equidistant from each of the three vertices. Problem 5 and 6 could wait till next day and teacher could give examples if they choose. These should also lead to a discussion of "prove," "verify," "counterexample," etc.

7. Find the equation of the line that is the perpendicular bisector of the segment with endpoints (-4, 4) and (6, 2). Students have had linear functions in grade 8. Thus they should be able to do this. Teacher may need to do a review of vocabulary.

12. Plane Geometry, circle

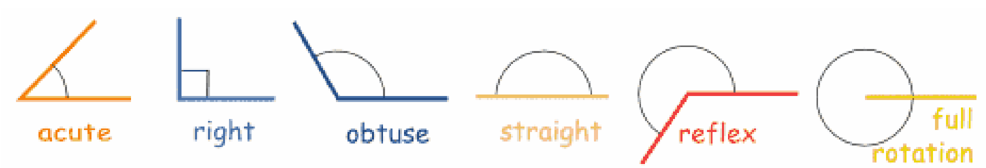
Plane Geometry

Plane geometry is all about shapes like lines, circles and triangles ... shapes that can be drawn on a flat surface called a **Plane** (it is like on an endless piece of paper).

If you like drawing, Geometry is for you!

General

Angles



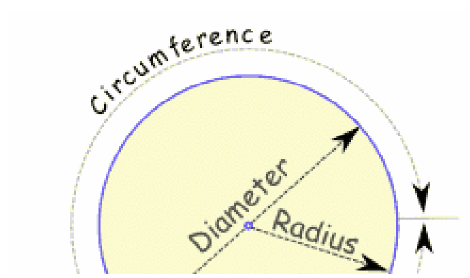
Circle

Here are some facts about circle.

Pi

Pi (the symbol is the Greek letter **π**) is:

The ratio of
the **Circumference**
to the **Diameter**
of a Circle.



In other words, if you measure the circumference, and then divide by the diameter of the circle you get the number **π**

It is approximately equal to:

3.14159265358979323846…

The digits go on and on with no pattern. In fact, **π** has been calculated to over one trillion decimal places and still there is no pattern.

Approximation

A quick and easy approximation to **π** is 22/7

22/7 = **3.1428571...**

But as you can see, 22/7 is **not exactly right**. In fact **π** is not equal to the ratio of any two numbers, which makes it an irrational number.

To 100 Decimal Places

Here is **π** with the first 100 decimal places:

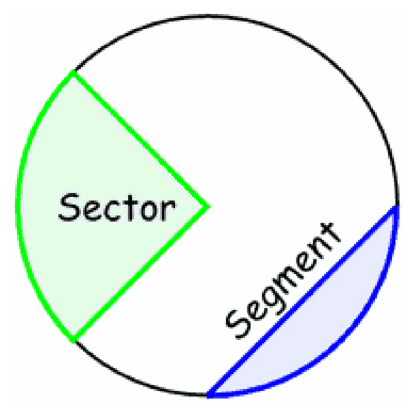
3.14159265358979323846264338327950288419716939937510
58209749445923078164062862089986280348253421170679...

Circle Sector and Segment

Slices

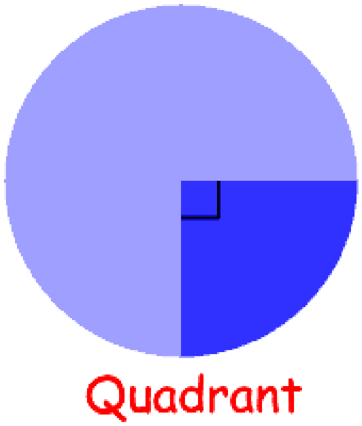
There are two main "slices" of a circle:

- The "pizza" slice is called a **Sector**.
- And the slice made by a chord is called a **Segment**.



Common Sectors

The Quadrant and Semicircle are two special types of Sector:



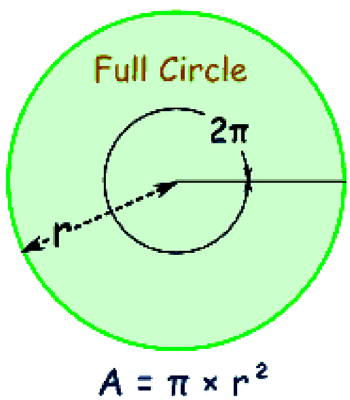
Quarter of a circle is called a **Quadrant**.

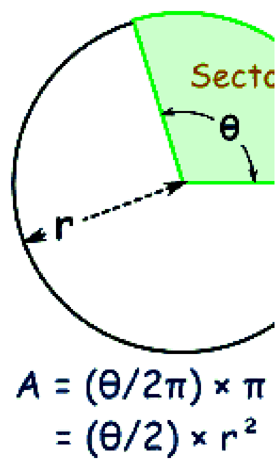
Half a circle is called a **Semicircle.**

Area of a Sector

You can work out the Area of a Sector by comparing its angle to the angle of a full circle.

Note: I am using radians for the angles.



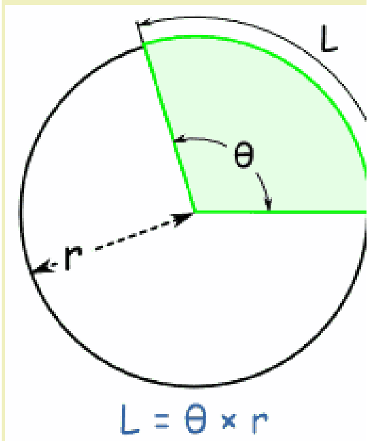


This is the reasoning:

- A circle has an angle of 2π and an Area of: πr^2
- So a Sector with an angle of θ (instead of 2π) must have an area of: $(\theta/2\pi) \times \pi r^2$
- Which can be simplified to: $(\theta/2) \times r^2$

Area of Sector = ½ × θ × r^2 *(when θ is in radians)*

Area of Sector = ½ × (θ × **π**/180) × r^2 *(when θ is in degrees)*



Arc Length

By the same reasoning, the arc length (of a Sector or Segment) is:

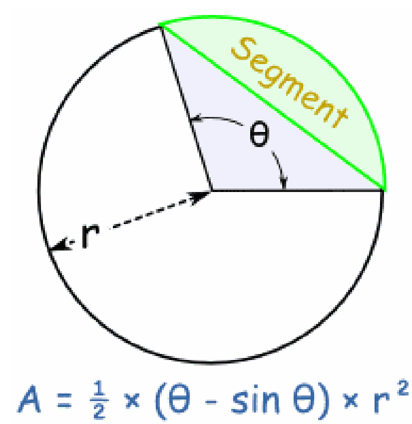
L = θ × r *(when θ is in radians)*

L = (θ × **π**/180) × r *(when θ is in degrees)*

Area of Segment

The Area of a Segment is the area of a sector minus the triangular piece (shown in light blue here).

There is a lengthy reason, but the result is a slight modification of the Sector formula:



Area of Segment = ½ × (θ - sin θ) × r^2 *(when θ is in radians)*

Area of Segment = ½ × ((θ × **π**/180) - sin θ) × r^2 *(when θ is in degrees)*

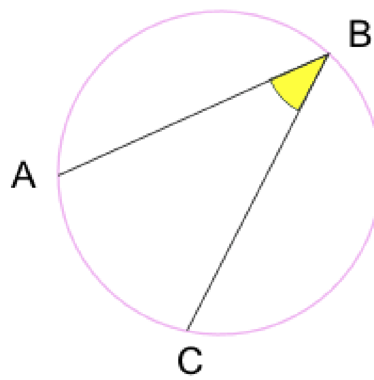
Circle Theorems

There are some interesting things about angles and circles that I want to share with you:

Inscribed Angle

First off, a definition:

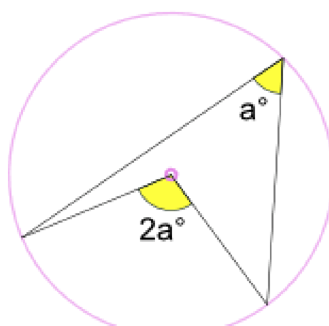
Inscribed Angle: an angle made from points sitting on the circle's circumference.



A and C are "end points"
B is the "apex point"

Inscribed Angle Theorems

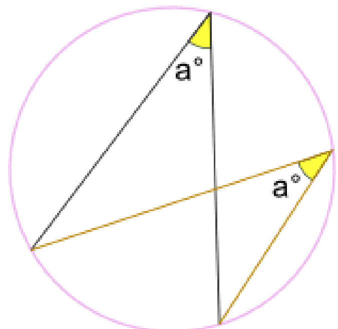
An inscibed angle a° is half of the central angle 2a°



(Called the **Angle at the Center Theorem**)

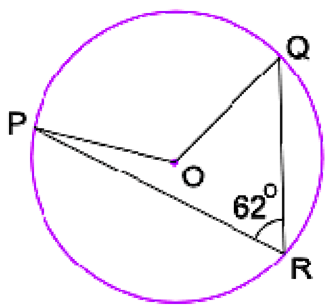
And (keeping the endpoints fixed) ...

... the angle a° is **always the same**, no matter where it is on the circumference:



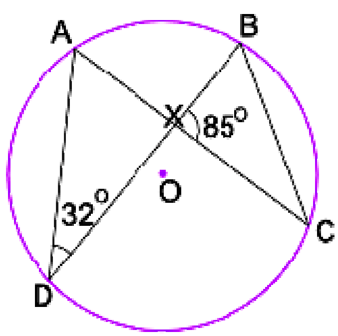
Angle a° is **the same**.
(Called the **Angles Subtended by Same Arc Theorem**)

Example: What is the size of Angle POQ? (O is circle's center)



Angle POQ = 2 × Angle PRQ = 2 × 62° = **124°**

Example: What is the size of Angle CBX?



Angle ADB = 32° is the same angle as Angle XCB

Now use angles of a triangle add to 180° in triangle BXC

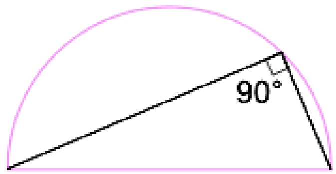
Angle CBX + Angle BXC + Angle XCB = 180°

Angle CBX + 85° + 32° = 180°

Angle CBX = 63°

Angle in a Semicircle

An angle **inscribed** in a **semicircle** is always a right angle:

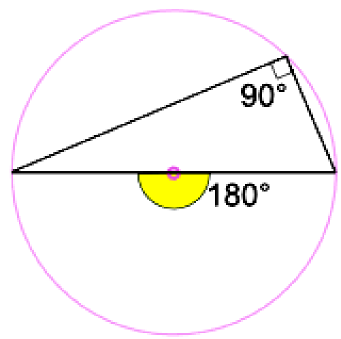


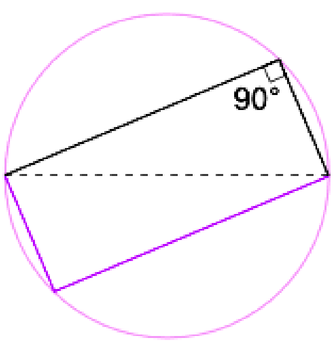
(The end points are either end of a circle's diameter,
the apex point can be anywhere on the circumference.)

Why? Because:

The inscribed angle 90° is half of the central angle 180°

(Using "Angle at the Center Theorem" above)



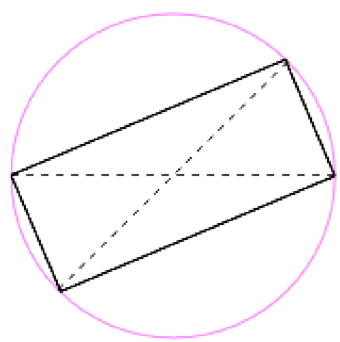


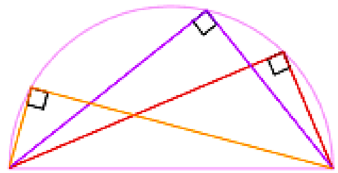
Another Good Reason Why It Works

We could also rotate the shape around 180° to make a rectangle!

It **is** a rectangle, because all sides are parallel, and both diagonals are equal.

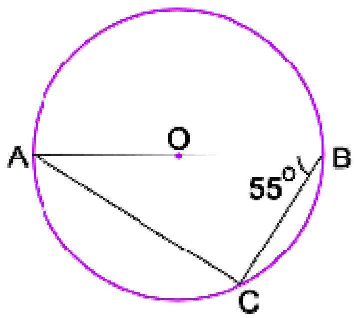
And so its internal angles are all right angles (90°).





So there you go! No matter **where** that angle is on the circumference, it is **always 90°**

Example: What is the size of Angle BAC?



The Angle in the Semicircle Theorem tells us that Angle ACB = 90°

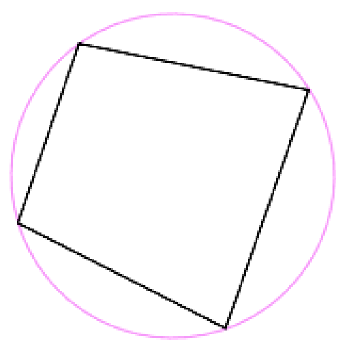
Now use angles of a triangle add to 180° to find Angle BAC:

Angle BAC + 55° + 90° = 180°

Angle BAC = 35°

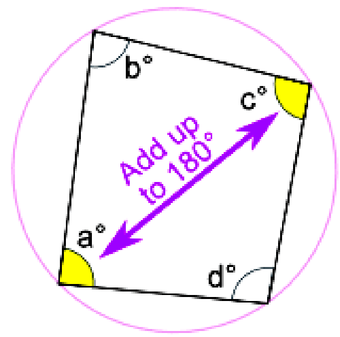
Cyclic Quadrilateral

A "Cyclic" Quadrilateral has every vertex on a circle's circumference:

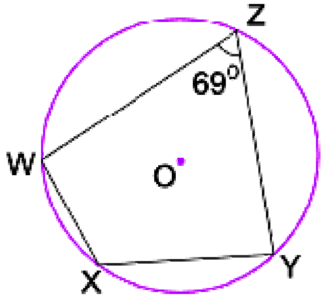


A Cyclic Quadrilateral's **opposite angles add to 180°**:

- a + c = 180°
- b + d = 180°



Example: What is the size of Angle WXY?

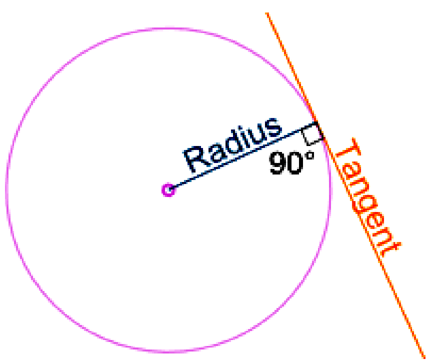


Opposite angles of a cyclic quadrilateral add to 180°

Angle WZY + Angle WXY = 180°

69° + Angle WXY = 180°

Angle WXY = 111°



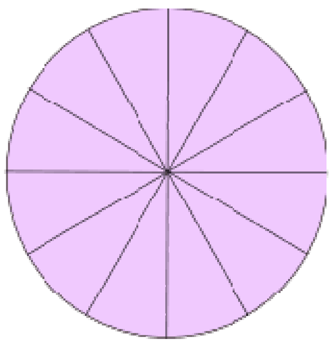
Tangent Angle

A tangent is a line that just touches a circle at one point.

It always forms a right angle with the circle's radius, as shown:

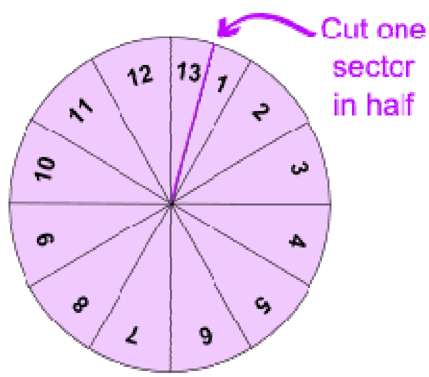
Area of a Circle by Cutting into Sectors

Here is a way to find the formula for the area of a circle:

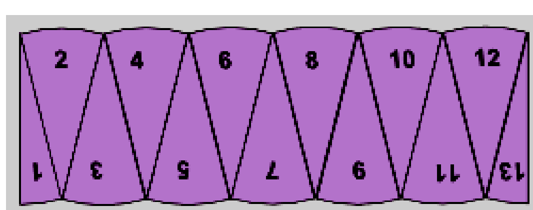


Cut a circle into equal sectors (12 in this example)

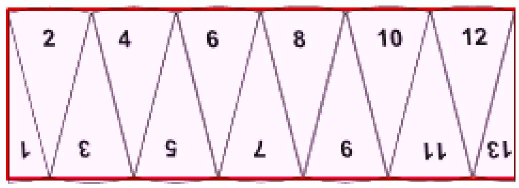
Divide just one of the sectors into two equal parts. You now have thirteen sectors – number them 1 to 13:



Rearrange the 13 sectors like this:



Which resembles a rectangle:



What are the (approximate) height and width of the rectangle?

The **height** is the circle's **radius**: just look at sectors 1 and 13 above. When they were in the circle they were "radius" high.

The **width** (actually one "bumpy" edge) is half of the curved parts along the circle's edge ... in other words it is about **half the circumference** of the circle.

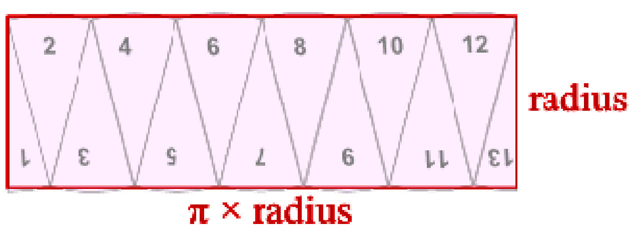
We know that:

$$\text{Circumference} = 2 \times \pi \times \text{radius}$$

And so the width is about:

$$\text{Half the Circumference} = \pi \times \text{radius}$$

And so we have (approximately):



Now we just multply the width by the height to find the area of the rectangle:

$$\text{Area} = (\pi \times \text{radius}) \times (\text{radius})$$

$$= \pi \times \text{radius}^2$$

Note: The rectangle and the "bumpy edged shape" made by the sectors are not an exact match.

But we could get a better result if we divided the circle into 25 sectors (23 with an angle of 15° and 2 with an angle of 7.5°).

And the more we divided the circle up, the closer we would get to being exactly right.

Conclusion

$$\text{Area of Circle} = \pi r^2$$

Activity: Garden Area

Have you ever wondered what the area of your garden is?

Let us try and find out!

You will need a garden, a tape measure, pen and paper ... and your brains.

I don't have a garden, so what can I do?

If you don't have a garden, I'm sure a friend has one, or your relatives have one, so use theirs.

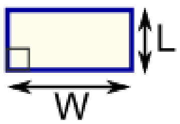
How accurate should my measurements be?

Try to measure to the nearest centimeter (or half-inch), so the error will be as small as possible.

You should get a good estimate, as long as you are careful with your measuring.

Is there a simple way to find the area of my garden?

If your garden is a rectangle, then you have a simple calculation. You just have to measure its width and length and multiply them together:



Rectangle: Area = W × L

- W = width
- L = length

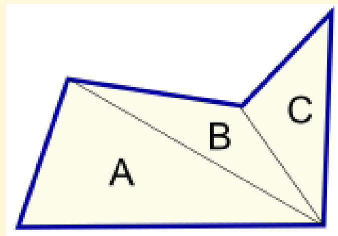
But that makes this activity just too easy ...
... so go find another garden with a more interesting shape!

My garden is a difficult shape, so how can I find its area?

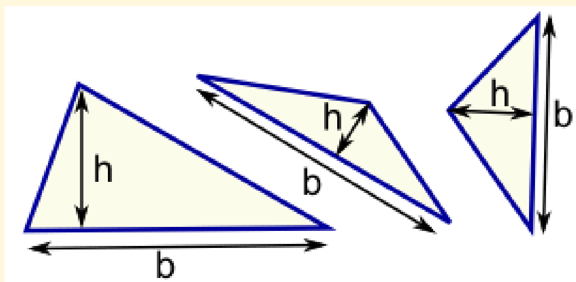
Good! This activity just got ***interesting*** ...

It may be one of the shapes on the page Area of Plane Shapes, then you just have to decide which shape, make the measurements, and use the formula.

But you could also break up your difficult shape into triangles:



Then measure the base (**b**) and height (**h**) of each triangle:



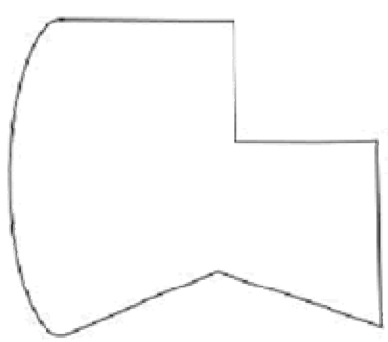
Write down each measurement carefully so you know which triangle it belongs to.

Now go inside and calculate each area (using Area = ½b × h) and add them all up.

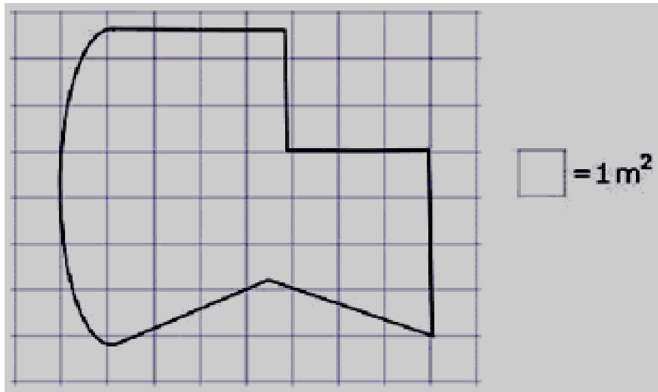
But my garden is different ...

... in fact it's not any shape at all ... it has some straight edges and some curved parts. What should I do?

Maybe it looks something like this:



You could try covering your garden with a grid of squares – these could be 1 metre squares or 1 foot squares, something like this:



How do I make a grid?

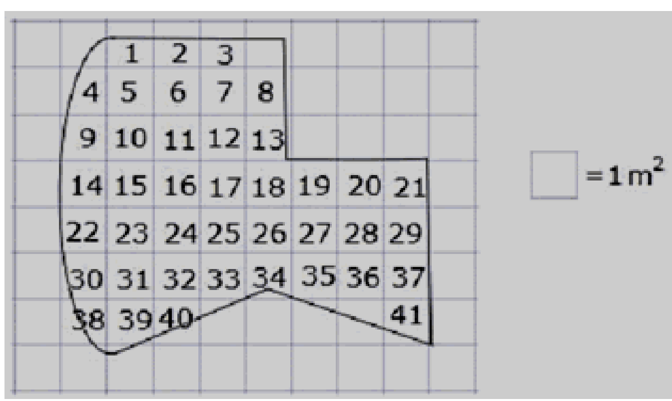
Try using pegs in the ground and join them up with string. Make sure they are the right distance apart and all angles are right angles.

How does this help? The grid and the outline of the garden don't match. There are lots of corners and curved parts.

Count the squares!

There are special methods talked about on the Area page. The simplest method is:

- more than half a square counts as 1
- less than half a square counts as 0



An estimate for this area is **41 m^2**. This is just an example. Your garden will be different.
(If your grid was 1 foot, then the area will be in square feet)

Why should I want to know the area of my garden?

There might be lots of reasons:

- You want to re-turf the garden. How much grass should you order? How much will it cost?
- You want to plant the garden with tomato plants. These have to be planted a certain distance apart. How many plants could you plant? What will be your expected yield of tomatoes?
- You want to hold a barbeque party. How many people could comfortably fit into your garden?

You can now do Activity: Grass for the Garden as a project work (as an Example)

Project work

Grass for the Garden

You want to re-turf the garden

Let us try and find out what it will cost!

Your first step is to find the area. The Garden Area Activity shows you how.

Area of Garden m^2 (or ft^2):

Now, you have two choices:

- buy **seed** (and enjoy watching the grass grow), or
- buy **turf** (and get instant weed-free results)?

Let us work out the cost of each.

Note: I include sample values, but you should use your own garden area, and also find out the cost of seed and turf where you live.

Seed

How much grass seed should I order?

You've already calculated the area of your garden, so now you will have to find out how much seed will cover each square meter, or each square foot.

This might also depend on which kind of grass seed you want to use, so you will have to decide that first.

You could visit your local garden center or farm supply store and read the instructions on the seed packet, or ask the store keeper for advice. Write down your choice here:

Type of Grass Seed:
How Much Seed per m^2 (or ft^2):

While you are at the store, collect some prices.

Packet Size:			
Cost:			

Now you've got all the information, what kind of calculation will you need to do?

Example: The area of my garden is $41m^2$.

I found out from the local garden centre that I will need 55g of seed for each m^2.

So the total amount of seed I will need is:

$$41 \times 55g = 2{,}255g$$

But the area calculation was only a rough estimate, so maybe I need to buy, say, 5% extra.

That makes $2{,}255g \times 105\% = 2{,}368g$

Or about **2.4 kg**

Tip: Always order a little more than you need!

Just in case your estimate is wrong, or some gets wasted.

Suppose the garden center only sells seed in 500g or 1kg packets. How many packets will I need to buy, and which way will be the most economical?

I could buy:

- Five 500g packets
- Three 500g packets and one 1kg packet
- One 500g packet and two 1kg packets

Which do you think would be the cheapest way of doing it?

Maybe the third option would be the cheapest. It's usually more economical to buy larger quantities.

But watch out! Sometimes they might sell the smaller quantity at a special rate as a special promotion. Also check out the prices of different brands to get the best bargain.

Too much?

But won't that be too much seed?

Yes, of course you will have some seed left over (unless you are very lucky), but you can keep some in case you need to reseed part of your lawn another time, or in case the birds ate some of the seed!

How much will it cost?

Once you know how many packets of seed you require and what sizes, you can calculate the cost of the seed. That's easy.

Example: A 500g packet of seed costs Rs 200 and a 1kg packet costs Rs. 350.

I need one 500g packet, so 1 × 200 =		200
I need two 1kg packets, so 2 × 350 =		700
The total cost is:		**900**

Your Turn:

Packet Size	Quantity	Cost Per Packet	Total Cost
		Grand Total:	

Note: you might have to do the calculations using square feet, and find out the costs of different sizes of packet in 1lb packets and 2lb packets. But the idea's the same.

Turf

Some people prefer to use turf instead of seed. It may be easier and quicker to lay, but will it cost more?

The calculation should be fairly simple. First of all, decide which kind of turf you need.

Then all you need to know is the cost of $1m^2$ of turf (or $1ft^2$ of turf). Again, you should be able to find out this information from your local garden center or farm supply store.

Example: I found out from the local garden center that turf costs Rs. 300 per m^2

I have an area of $41m^2$, but this was only a rough estimate, and I may lose some when cutting it to shape, say 10% extra.

$$\text{That's } 41m^2 \times 110\% = 45m^2$$

Then the cost of turfing my garden would be 45×300 = Rs. 13,500

For me it costs about twice as much as seeding, but this is just an example. Try it for your own garden.

Turf Area	Cost per Area	Total Cost

Note: you might have to do the calculations using square feet, and find out the cost per $1ft^2$ of turf.

Now you know the cost of both options, which do you chose?

Your Choice:

I am finished ... what have I learned?

You have learned about measuring, recording data, drawing, and calculating area, well done!

13. Trigonometric graphs

Why study trigonometric graphs?

The graphs in this section are probably the most commonly used in all areas of science and engineering. They are used for **modeling** many different natural and mechanical phenomena (populations, waves, engines, acoustics, electronics, UV intensity, growth of plants and animals, etc).

The trigonometric graphs in this chapter are **periodic**, which means the shape repeats itself exactly after a certain amount of time. Anything that has a **regular cycle** (like the tides, temperatures, rotation of the earth, etc) can be modeled using a sine or cosine curve.

In this chapter...

1. Graphs of $y = a \sin x$ and $y = a \cos x$, talks about **amplitude.** Amplitude is a indication of how much energy a wave contains.
2. Graphs of y = a sin bx and y = a cos bx introduces the **period** of a trigonometric graph.
3. Graphs of y = a sin(bx+c) and y = a cos(bx+c) helps you to understand the **displacement** (or **phase shift**) of a trigonometric curve.
4. Graphs of tan x, cot x, sec x and csc x are not as commonly used in the study of periodic activity, but are used in some applications.
5. Applications of Trigonometric Graphs includes the interesting What are the frequencies of music notes?.
6. Composite Trigonometric Curves arise when we add more than one waveform.
7. Lissajous Figures are a special kind of composite trigonometric graph.

Overview of Trigonometric Graphs

Here's a movie that gives an overview of the concepts in this chapter.

We begin the chapter with an examination of what amplitude means and the effect of the "a" variable in 1. Graphs of y = a sin x and y = a cos x »

The Sine Curve $y = a \sin t$

Let's investigate the shape of the curve $y = a \sin t$ and see what the concept of "**amplitude**" means. The sine curve occurs naturally when we are examining waves.

Have a play with the following Flash interactive. Run the animation first (click "Start"). Then change the circle radius (which changes the amplitude of the sine curve) using the slider. Then run the animation again.

The scale for this is **radians**. Remember that π radians is 180°, so in the graph, the value of 3.14 on the t-axis represents 180° and 6.28 is equivalent to 360°.

Did you notice:

- That the shape of the sine curve forms a **regular pattern** (the curve repeats after the wheel has gone around once)? We say such curves are **periodic**. The **period** is the time it takes to go through one cycle and then start over again.
- That in the interactive, when the radius of the circle was 50 units then the curve went up to 50 units and down to -50 units on the y-axis? This quantity of a sine curve is called the **amplitude** of the graph. This indicates how much **energy** is involved in the quantity being graphed. Higher amplitude means greater energy.
- That the rotation angle in **radians** is the same as the time (in seconds, well approximately). See more on radians. All the graphs in this chapter deal with angles in radians. Radians are much more useful in engineering and science than degrees.

Amplitude

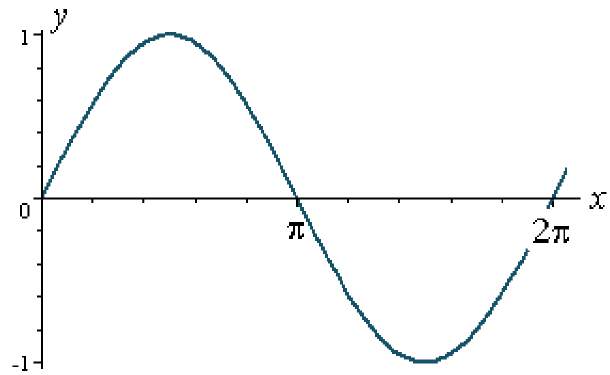
The a in the expression $y = a \sin x$ represents the **amplitude** of the graph. It is an indication of how much **energy** the wave contains.

The amplitude is the distance from the "resting" position (otherwise known as the **mean value** or **average value**) of the curve. In the interactive above, the amplitude can be varied from 10 to 100 units.

Amplitude is always a **positive** quantity. We could write this using absolute value signs. For the curves $y = a \sin x$,

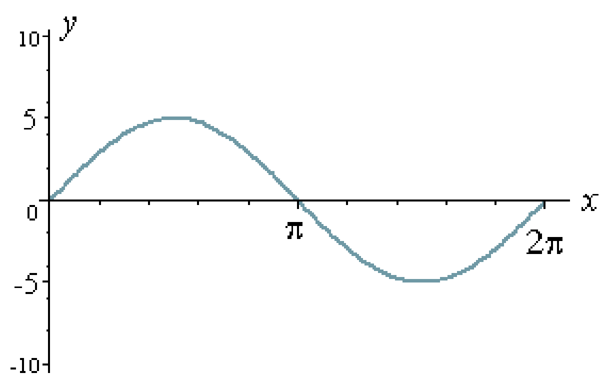
Amplitude = $|a|$

Graph of Sine *x* - with varying amplitudes



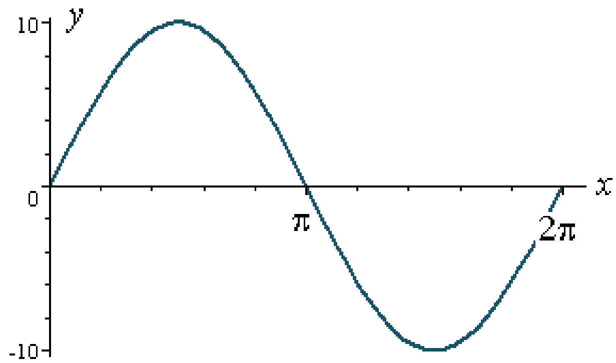
We start with $y = \sin x$.

It has **amplitude** = 1 and **period** = 2π.



Now let's look at the graph of $y = 5 \sin x$.

This time we have **amplitude** = 5 and **period** = 2π. (I have used a different scale on the y-axis.)



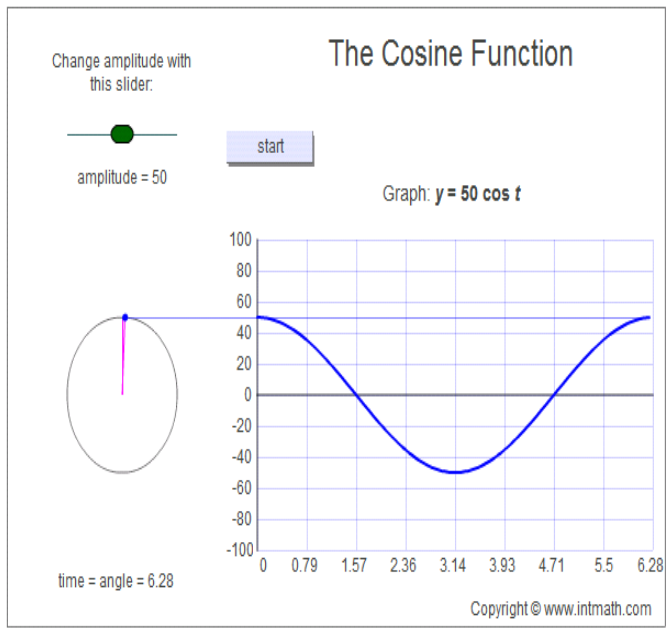
And now for $y = 10 \sin x$.

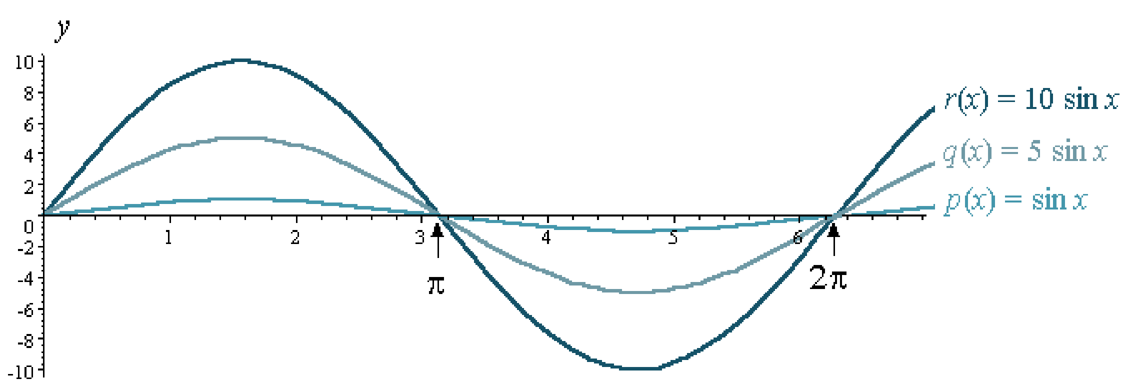
Amplitude = 10

and **period** = 2π.

For comparison, and using the same y-axis scale, here are the graphs of $p(x) = \sin x, q(x) = 5 \sin x$ and $r(x) = 10 \sin x$ on the one set of axes.

Note that the graphs have the same **period** (which is 2π) but different **amplitude**.





Graph of Cosine *x* - with varying amplitudes

Now let's see what the graph of $y = a \cos x$ looks like.

Similar to the sine interactive at the top of the page, you can change the amplitude using the slider.

Did you notice ?

- That the sine and cosine graphs are almost identical, except the cosine curve is shifted to the left by π/2 (= 1.57 = 90°)?

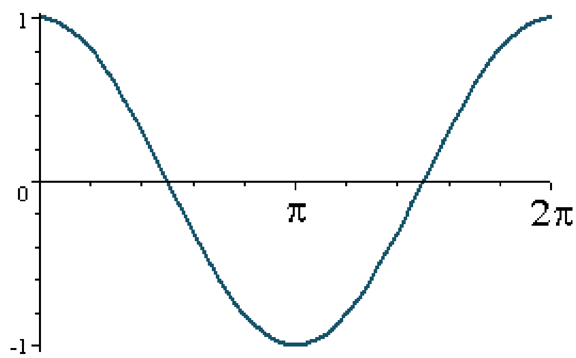
Now let's have a look at the graph of $y = \cos x$.

We note that the **amplitude** = 1 and **period** = 2π.

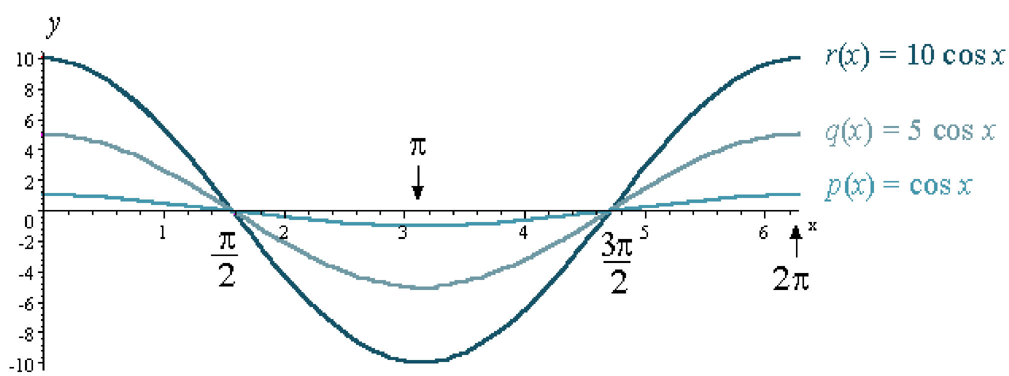
Similar to what we did with

$y = \sin x$ above, we now see the graphs of

- $p(x) = \cos x$
- $q(x) = 5 \cos x$
- $r(x) = 10 \cos x$

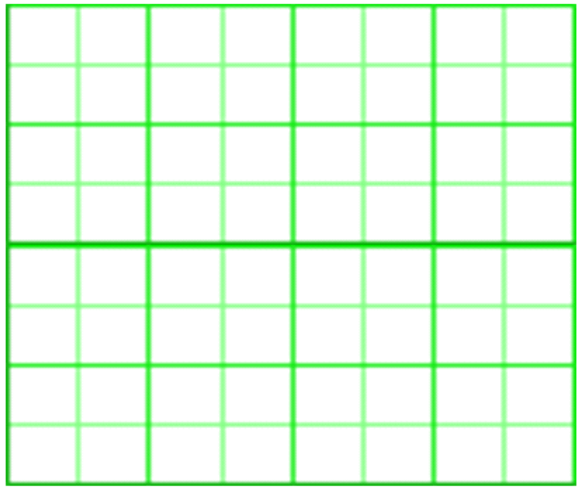


on one set of axes, for comparison:



Note: For the cosine curve, just like the sine curve, the **period** of each graph is the same (2π), but the **amplitude** has changed.

Exercises



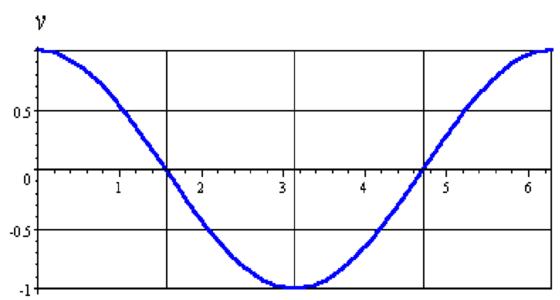
Sketch one cycle of the following **without** using a table of values! (The important thing is to know the **shape** of these graphs - not that you can join dots!)

Each one has period 2π. We learn more about period in the next section Graphs of y = a sin bx.

The examples use t as the independent variable. In electronics, the variable is most often t.

1) $i = \sin t$

Answer



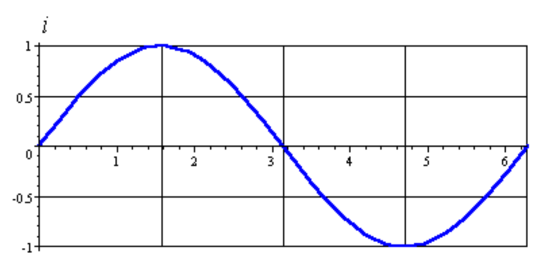
We saw this curve above, except now we are using i for current and t for time. These are very common variables in trigonometry.

Period = 2π

Amplitude = 1

2) $v = \cos t$

Answer

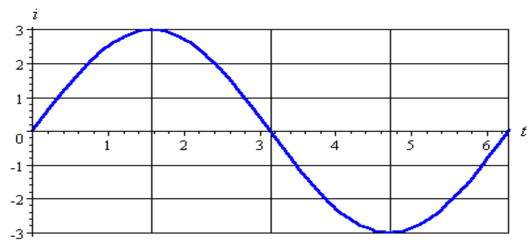


Once again, we saw this curve above, except now we are using v for voltage and $t$for time.

Period = 2π

Amplitude = 1

3) $i = 3 \sin t$



Answer

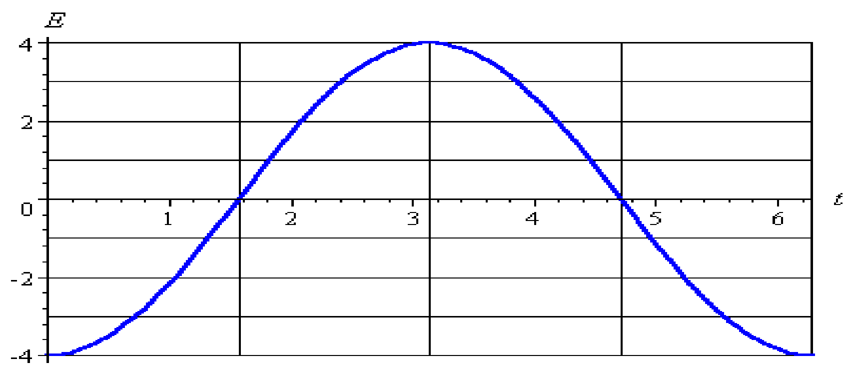
Period = 2π

Amplitude = 3

4) $E = -4 \cos t$

Answer

The variable E is used for "electro-motive force", another term for voltage.



Period = 2π

Amplitude = 4

Notice that:

- The negative in front of the cosine has the effect of turning the cosine curve "upside down". That is, it is a mirror image in the horizontal t axis.
- Amplitude is a positive number (it is a distance)

2. Graphs of y = a sinb x and y = a cos bx

by M. Bourne

The *b* in both of the graph types

- y = a sin bx
- y = a cos bx

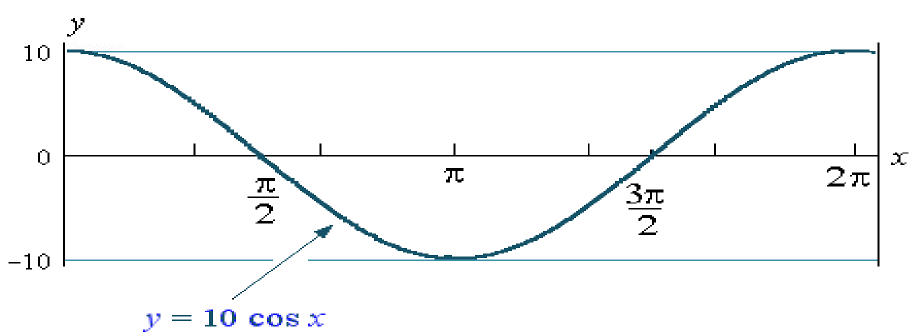
affects the period (or wavelength) of the graph. The period is the distance (or time) that it takes for the sine or cosine curve to begin repeating again.

The period is given by:

$$\text{Period} = \frac{2\pi}{b}$$

Note: As b gets larger, the period decreases.

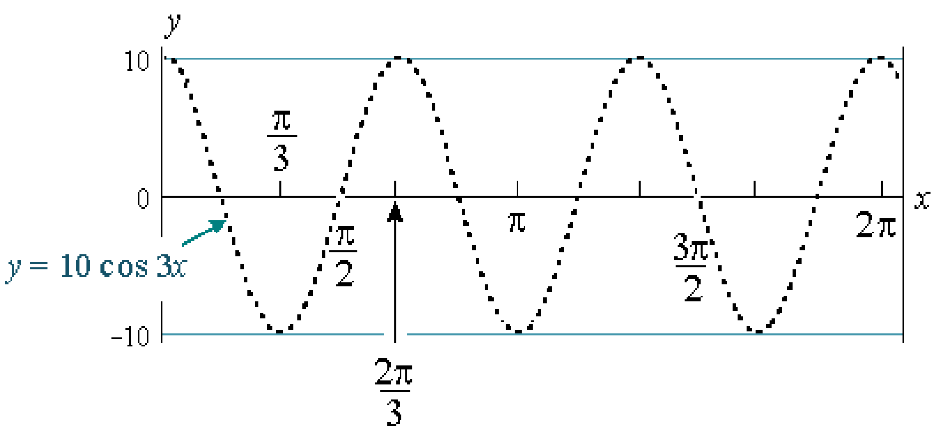
Changing the period



First, let's look at the graph of y=10 cos x, which we learned about in the last section,sine and cosine curves.

As we learned,the period is 2π

Now let's look at y = 10 cos 3x. Note the 3 inside the cosine term.



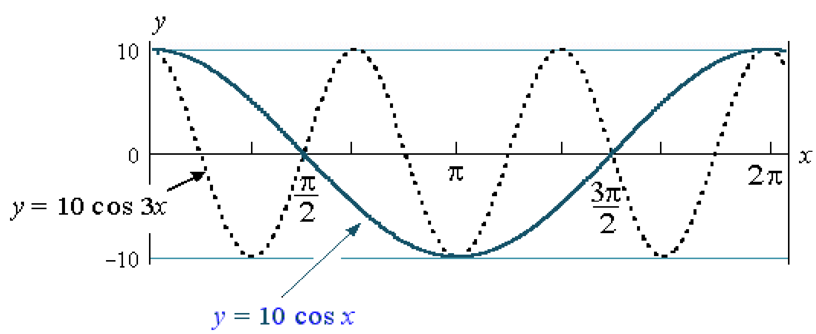
Notice that the period is different. (The amplitude is 10 in each example.)

This time the curve starts to repeat itself at x=2π/3.

This is consistent with the formula we met above :

$$\text{Period} = \frac{2\pi}{b}$$

Now let's view the 2 curves on the same set of axes. Note that both graphs have amplitude of 10 units, but their period is different.



Cosine graphs

Let's play with the graphs that we have just drawn. In this case, you can vary the period (and the amplitude) by using the sliders at the bottom.

You can also change the function to whatever you like.

Answer

Good to know……………….

Remember , we are now operating using RADIANS Recall that:

$2\pi = 6.283185…$

Remember, we are now operating using RADIANS. Recall that: 2π=3600

We only use radians here.

Note : b tells us the number of cycles in each 2π.

For y = 10cos x, there is one cycle between 0 and **2**π (because b=1).

For y = 10 cos 3x, there are 3 cycles between 0 and 2π (because b=3) .

Flash interactive – Pistons and the Period of a Sine Curve

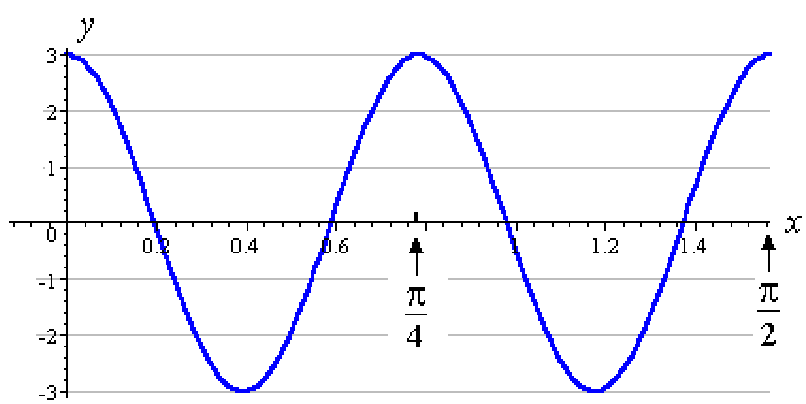
Here's another interactive that you can use to explore the concept of period and frequency. The frequency =1/period. We'll see more on this below:

The piston engine is the most commonly used engine in the world. Its motion can be described using a sine curve.

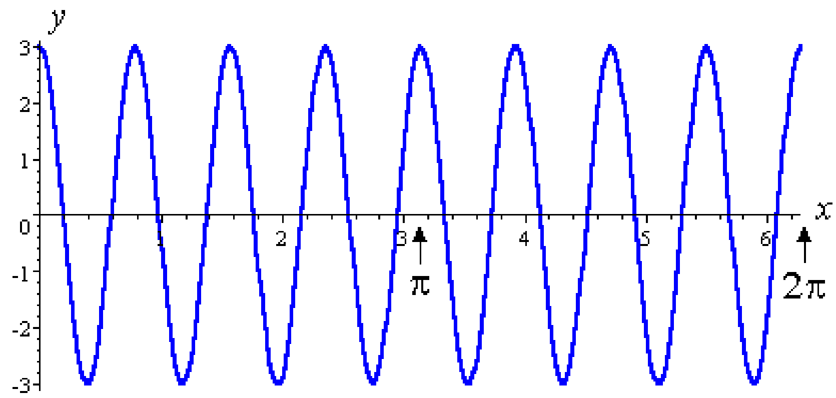
1. Sketch 2 cycles of y=3 cos 8x.

Answer

Here, $b = 8$, so the period is $2\pi/8 = \pi/4$. To draw 2 cycles, we will need to graph from 0 to π/2 along the x-axis.



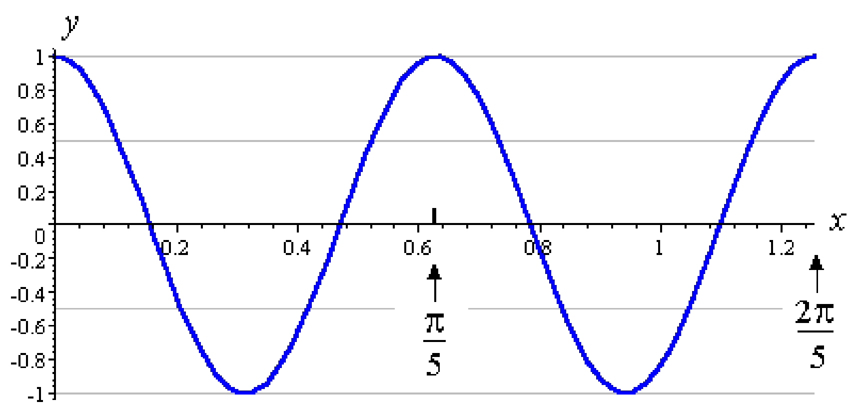
Now for interest, let's see what it looks like from 0 to 2π.



2. Sketch 2 cycles of y= cos 10x.

Answer

In this example, $b = 10$, so the period is $2\pi/10 = \pi/5$. To draw 2 cycles, we will need from 0 to $2\pi/5$ along the x-axis.

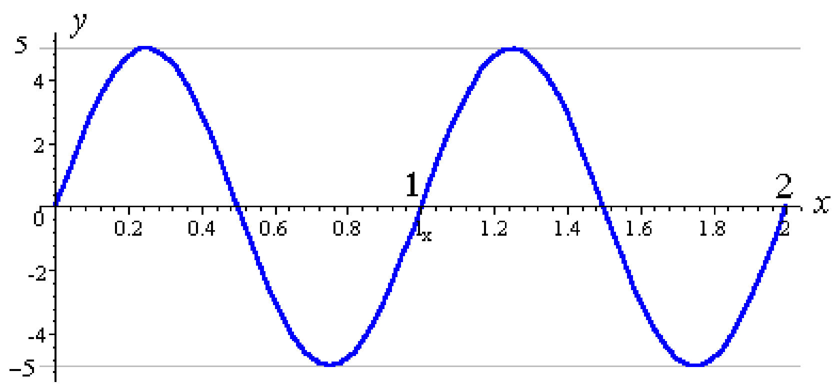


Note that one cycle has period $2\pi/10 = 0.628$ and there will be **10** cycles between 0 and 2π.

3. Sketch 2 cycles of y= 5 sin 2πx.

Answer

In this example, $b = 2\pi$. So the period is $2\pi/2\pi = 1$.

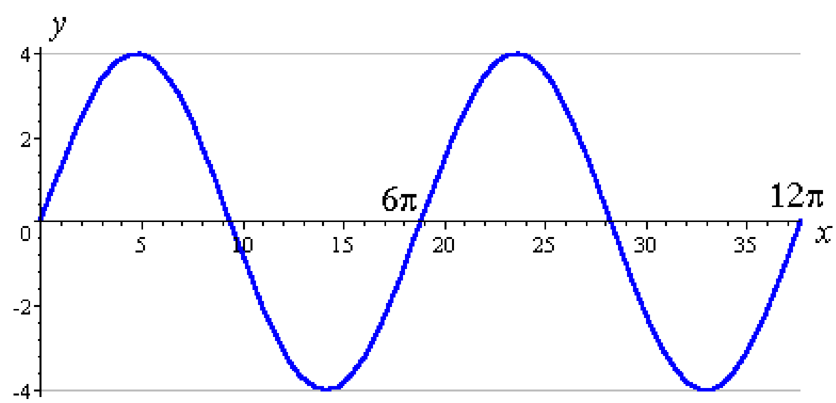


This time, we do not have any multiple of π in our horizontal scale.

4. Sketch 2 cycles of y= 4 sin x/3

Answer

In this example, b = 1/3, so the period is 6π = 18.85.



Defining Sine Curves using Frequency

It is common in electronics to express the sin graph in terms of the frequency f as follows:

Y= sin 2πft

This is very convenient, since we don't have to do any calculation to find the frequency (like we were doing above). The frequency, f , is normally measured in cycles / second, which is the same as Herz (Hz).

The period of the curve (the time it takes to go from one crest to the next crest) can be found easily once we know the frequency:

$$T = \frac{1}{f}$$

The units for period are normally seconds.

Example :

Household voltage in the UK is alternating current, 240V with frequency 50 Hz. What is the equation describing this voltage?

Answer

The voltage could be described as: $V = 240 \sin 2\pi(50)t$.

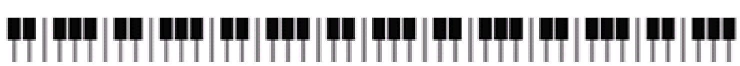
The period of the voltage is 1/50 = 0.02 seconds.

The graphs in this section are probably the most commonly used in all areas of science and engineering. They are used for **modeling** many different natural and mechanical phenomena (populations, waves, engines, acoustics, electronics, UV intensity, growth of plants and animals, etc).

The trigonometric graphs in this chapter are **periodic**, which means the shape repeats itself exactly after a certain amount of time. Anything that has a **regular cycle** (like the tides, temperatures, rotation of the earth, etc) can be modeled using a sine or cosine curve.

Music Example

The frequency of a note in music depends on the period of the wave. If the frequency is high, the period is short; if the frequency is low, the period is longer



Here is an
interesting question which a student asked me recently. She wanted to know the frequencies of all the notes on a piano.

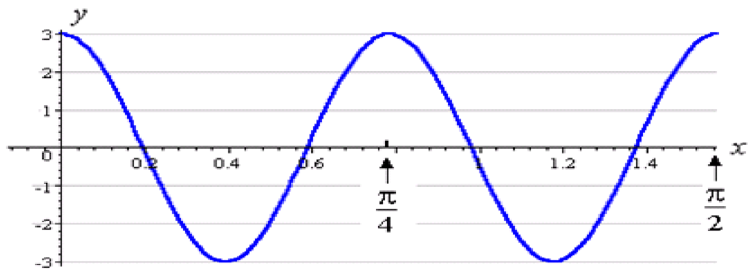
A piano is tuned to A =440 Hz (cycles/second) and the other notes are evenly spaced, 12 notes to each octave. A note an octave higher than A=440Hz has twice the frequency (880 Hz) and an octave lower than A = 440 Hz has half the frequency (220 Hz)

1. Sketch 2 cycles of $y = 3 \cos 8x$.

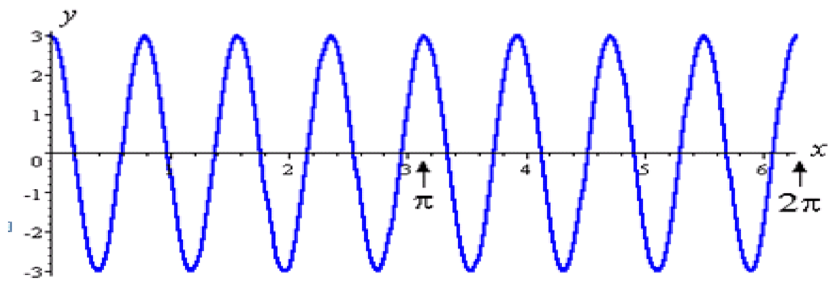
[You can use the Java applet above to help you understand how the sketch works.)

Answer

Here,b=8, so the period is 2π/8=π/4. To draw 2 cycles,we will need to graph from 0 to π/2 along the x-axis.



Now for interest , let's see what it looks like from 0 to 2π.



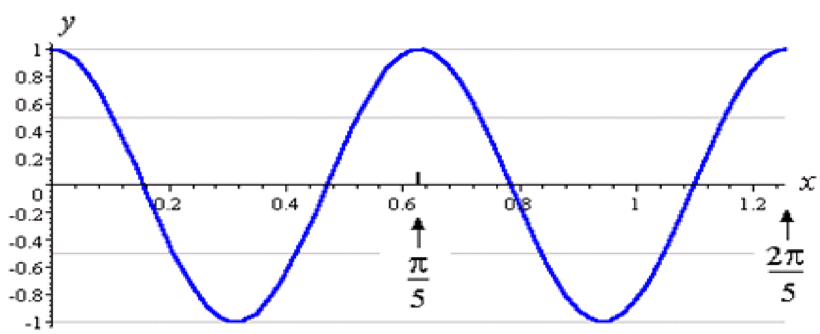
Note that there are 8 cycles between 0 and 2π.

Also, note that we started the graph at x=0, but we could have started anywhere. As long as we draw exactly 2 cycles, we are answering the question.

2. Sketch 2 cycles of y=cos10x.

Answer

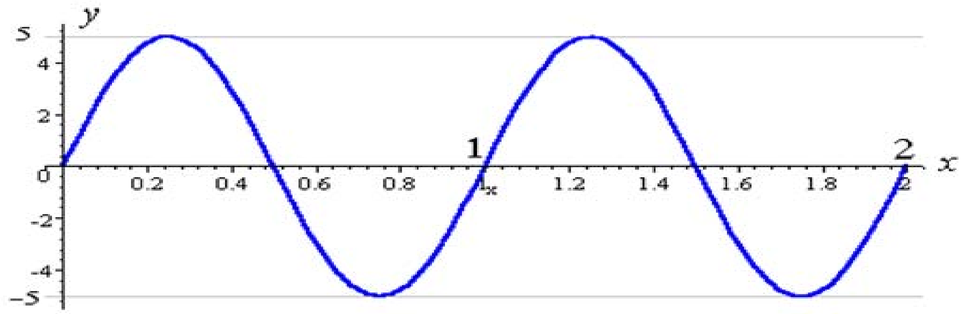
In this example, b=10, so the period is 2π /10= π/5.To draw 2 cycles, we will need from 0 to 2π/5 along the x-axis.



Note that one cycle has period 2π /10=0.628 and there will be 10 cycles between 0 and 2π.

3. Sketch 2 cycles of y=5sin2x.

Answer: In this example, b=2π, So the period is 2π/2π =1.

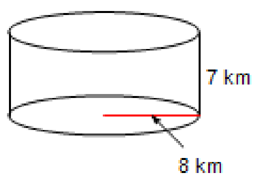


This time, we do not have any multiple of 2π in our horizontal scale.

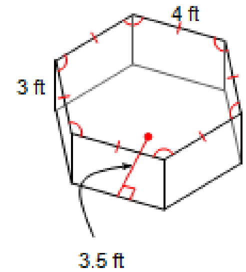
14. Some examples for practice

Find the volume of the given prisms and cylinders.

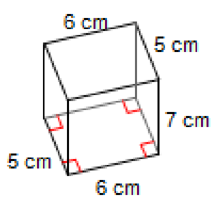
1)



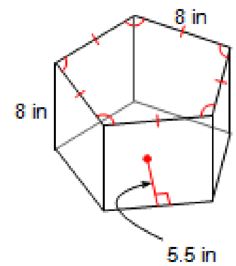
2)



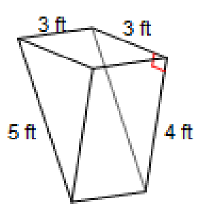
3)



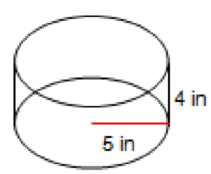
4)



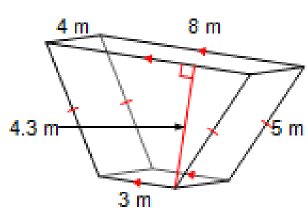
5)



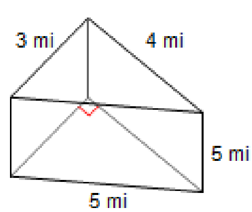
6)



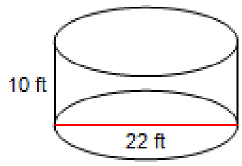
7)



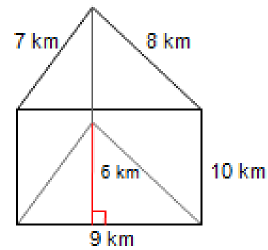
8)



9)



10)



11) A cylinder with a radius of 4 yd and a height of 5 yd.

12) A square prism measuring 6 km along each edge of the base and 5 km tall.

13) A hexagonal prism 5 yd tall with a regular base measuring 5 yd on each edge and an apothem of length 4.3 yd.

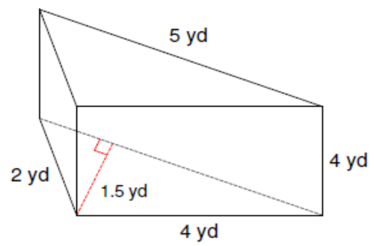
14) A trapezoidal prism of height 6 km. The parallel sides of the base have lengths 5 km and 3 km. The other sides of the base are each 2 km. The trapezoid's altitude measures 1.7 km.

Answers
1) 1407.4 km^3 2) 126 ft^3 3) 210 cm^3 4) 880 in^3
5) 314.2 in^3 6) 18 ft^3 7) 94.6 m^3 8) 30 mi^3
9) 3801.3 ft^3 10) 270 km^3 11) 251.3 yd^3 12) 180 km^3
13) 322.5 yd^3 14) 40.8 km^3

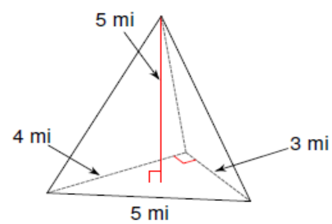
Volumes of Solids

Find the volume of each figure. Round to the nearest tenth.

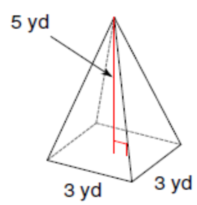
1)



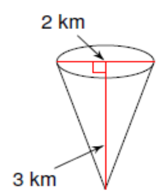
2)



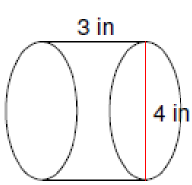
3)



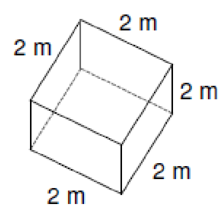
4)



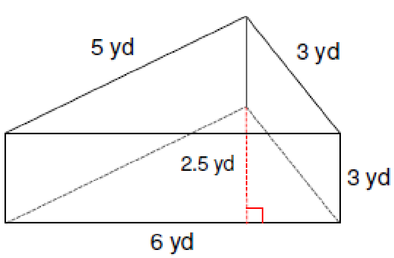
5)



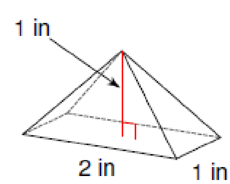
6)



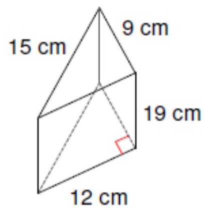
7)



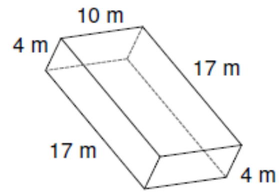
8)



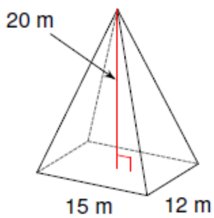
9)



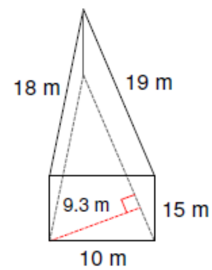
10)



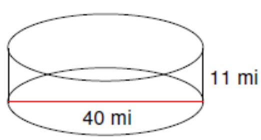
11)



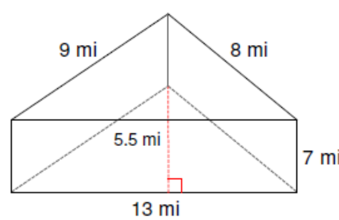
12)



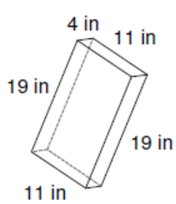
13)



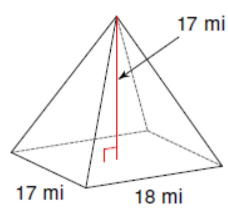
14)



15)



16)



17) A cylinder with a radius of 3 cm and a height of 7 cm.

18) A cone with diameter 20 cm and a height of 20 cm.

19) A cone with diameter 14 yd and a height of 14 yd.

20) A rectangular prism measuring 10 m and 7 m along the base and 12 m tall.

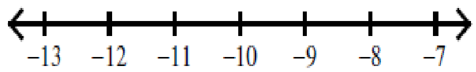
Answers

1) 15 yd^3 2) 10 mi^3 3) 15 yd^3 4) 3.1 km^3 5) 37.7 in^3
6) 8 m^3 7) 22.5 yd^3 8) 0.7 in^3 9) 1026 cm^3 10) 680 m^3
11) 1200 m^3 12) 1325.3 m^3 13) 13823 mi^3 14) 250.3 mi^3 15) 836 in^3
16) 1734 mi^3 17) 197.9 cm^3 18) 2094.4 cm^3 19) 718.4 yd^3 20) 840 m^3

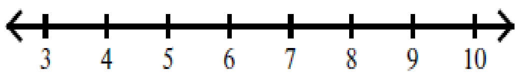
Solving Two-Step Inequalities

Solve each inequality and graph its solution.

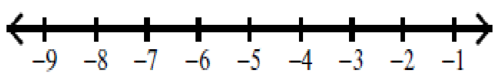
1) $5 + \frac{p}{9} \geq 4$



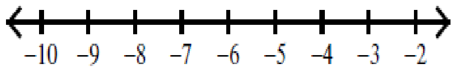
2) $-3k - 2 < -17$



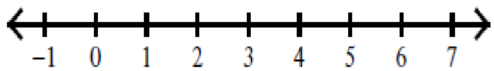
3) $-3 + \frac{k}{3} > -5$



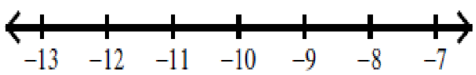
4) $3r + 3 \geq -12$



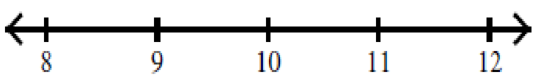
5) $\dfrac{8 + r}{4} \geq 3$



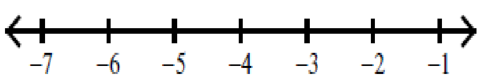
6) $-8k + 6 \leq 94$



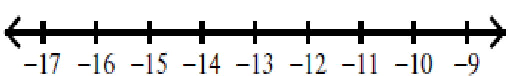
7) $-8 + 9x \geq 82$



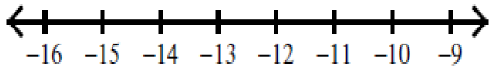
8) $\dfrac{n - 2}{3} > -2$



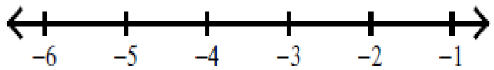
9) $7b + 8 \leq -83$



10) $-1 < \dfrac{-8+m}{22}$

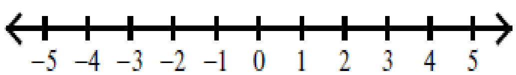


11) $-2 \le \dfrac{-6+n}{5}$

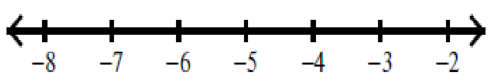


12) $4 < \dfrac{x-2}{2}$

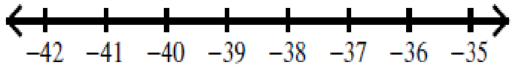
13) $-2n + 7 \ge 7$



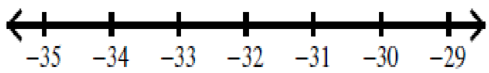
14) $-2 \le \dfrac{1+p}{2}$



15) $11 \geq 14 + \frac{p}{13}$

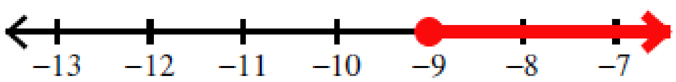


16) $-14 + \frac{p}{8} \geq -18$



Answer

1)



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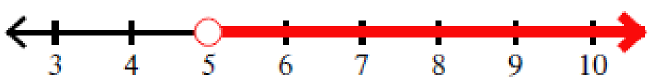

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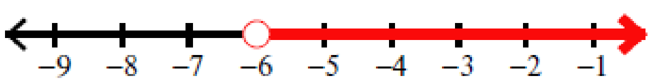

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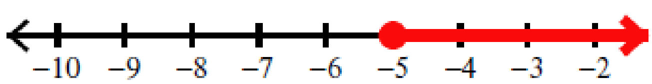
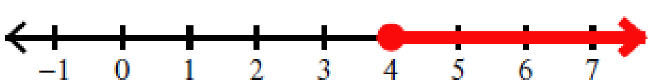
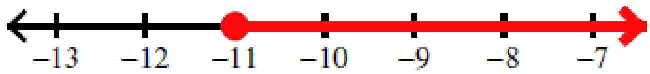


5)



6)



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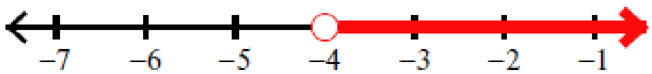

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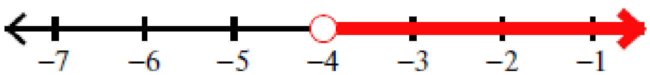
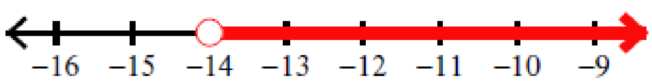
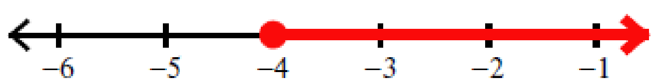


9)

10)



11)



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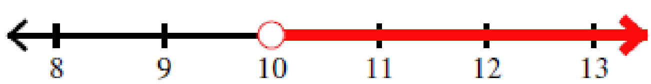
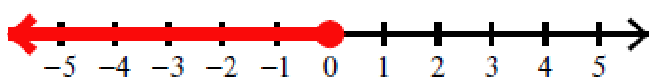


13)



14)

15)

16)

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